Complex surfaces

lecture 9: Atiyah-Singer' index theorem

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Fredholm operators

REMARK: All operators today **are considered continuous ("bounded").**
DEFINITION: A continuous operator $F: H_1 \longrightarrow H_2$ of Hilbert spaces is called **Fredholm** if its image is closed and kernel and cokernel are finite-dimensional.
REMARK: An injective operator with closed image **is invertible by Banach-Schauder theorem.**
DEFINITION: An operator $F: H_1 \longrightarrow H_2$ **has finite rank** if its image is finite-dimensional.
CLAIM: An operator $F: H_1 \longrightarrow H_2$ is Fredholm if and only if there exists $F_1: H_2 \longrightarrow H_1$ such that **the operators $\operatorname{Id} - FF_1$ and $\operatorname{Id} - F_1F$ have finite rank.**
Proof: Indeed, F defines an isomorphism $F: H_1/\ker F \longrightarrow \operatorname{im} F$ (Banach-Schauder). ■

DEFINITION: **Index** $\operatorname{ind}(F)$ of a Fredholm map F is the number $\dim\ker F - \dim\operatorname{coker} F$.

EXERCISE: Prove that **the index of a Fredholm operator is compatible with the composition,** $\operatorname{ind}(FG) = \operatorname{ind} F + \operatorname{ind} G$.

EXERCISE: Let F be Fredholm, and K a finite rank operator. **Prove that $\operatorname{ind}(F) = \operatorname{ind}(F+K)$.**

Small deformations of Fredholm operators are Fredholm

THEOREM: The set of Fredholm operators is open in the operator norm topology.

Proof. Step 1: Let $F: U \longrightarrow V$ be a Fredholm operator, and $U_1 := (\ker F)^\perp$. Since F is invertible on U_1, it satisfies $\inf_{x \in U_1} \frac{|F(x)|}{|x|} > 2\varepsilon$. Then, for any operator A with $\|A\| < \varepsilon$, one has $\inf_{x \in U_1} \frac{|F+A(x)|}{|x|} > \varepsilon$. **This implies that $F\Big|_{U_1}$ is an invertible map to its image, which is closed.** In particular, $\ker(F+A)$ is finite-dimensional.

Step 2: To obtain that $\operatorname{coker}(F+A)$ is finite-dimensional for $\|A\|$ sufficiently small, we observe that $\operatorname{coker}(F+A) = \ker(F^* + A^*)$, and F^* is also Fredholm. Then Step 1 implies that $\ker(F^* + A^*)$ **is finite-dimensional for $\|A\|$ sufficiently small.** ■

Small deformations of Fredholm operators have the same index

THEOREM: The function $F \longrightarrow \inf(F)$ **is locally constant in operator topology.**

Proof: Let $F, G: \ H_1 \longrightarrow H_2$ be Fredholm operators on a Hilbert spaces. Take F_1 such that $FF_1 = \operatorname{Id}_H + K$ and K has finite rank. This gives $GF_1 = \operatorname{Id}_H + K + (G - F)F_1$. The operator $\operatorname{Id}_H + (G - F)F_1$ is invertible for $|(G - F)| < |F_1|^{-1}$, hence it has index 0, and $\operatorname{Id}_H + (G - F)F_1 + K$ **has the same index, because index does not change if adding a finite rank operator.** Therefore, $\operatorname{ind}(GF_1) = \operatorname{ind}(FF_1)$. Using $\operatorname{ind}(AB) = \operatorname{ind} A + \operatorname{ind} B$, for $B = F_1$ and $A = F$ or G, we obtain $\operatorname{ind}(F) = \operatorname{ind} G$. ■

Sobolev's L^2-norm on $C^\infty_c(\mathbb{R}^n)$

DEFINITION: Denote by $C^\infty_c(\mathbb{R}^n)$ the space of smooth functions with compact support. For each differential monomial

$$P_\alpha = \frac{\partial^{k_1}}{\partial x_1^{k_1}}\frac{\partial^{k_2}}{\partial x_2^{k_2}}...\frac{\partial^{k_n}}{\partial x_1^{k_n}}$$

consider the corresponding partial derivative

$$P_\alpha(f) = \frac{\partial^{k_1}}{\partial x_1^{k_1}}\frac{\partial^{k_2}}{\partial x_2^{k_2}}...\frac{\partial^{k_n}}{\partial x_1^{k_n}}f.$$

Given $f \in C^\infty_c(\mathbb{R}^n)$, one defines **the L^2_p Sobolev's norm $|f|_p$** as follows:

$$|f|_s^2 = \sum_{\deg P_\alpha \leqslant p} \int |P_\alpha(f)|^2 \operatorname{Vol}$$

where the sum is taken over all differential monomials P_α of degree $\leqslant p$, and $\operatorname{Vol} = dx_1 \wedge dx_2 \wedge ...dx_n$ - the standard volume form.

L^2_p-topology and elliptic operators

DEFINITION: Let F be a vector bundle on a compact manifold. The **L^2_p-topology** on the space of sections of F is a topology defined by the norm $|f|_p$ with $|f|_p^2 = \sum_{i=0}^p \int_M |\nabla^i f|^2 \operatorname{Vol}_M$, for some connection and scalar product on F and $\Lambda^1 M$.

From now on, **we write (x, y) instead of $\int_M (x, y) \operatorname{Vol}_M$.** This metric is also denoted L^2; the space of sections of B with this metric (B, L^2).

DEFINITION: We define the **Sobolev's L^2_p-metric on vector bundles** by $L^2_p(x, y) = \sum_{i=0}^p (\nabla^i(x), \nabla^i(y))$.

This result lies in foundation of the theory of elliptic operators; we won't prove it.

DEFINITION: Let B_1, B_2 be vector bundles, and $D: B_1 \longrightarrow B_2$ a differential operator of order k. It is called **elliptic** if its symbol $\operatorname{symb}(D) \in \operatorname{Sym}^k TM \otimes \operatorname{Hom}(B_1, B_2)$, considered as a section of $\operatorname{End}(B)$ over T^*M, homogeneous of degree k, is invertible outside of the zero section.

THEOREM: Let B_1, B_2 be vector bundles, and $D: B_1 \longrightarrow B_2$ an elliptic differential operator of order p. **Then $D: (B_1, L^2_p) \longrightarrow (B_2, L^2)$ is Fredholm.**

REMARK: We will use this result only when $B_1 = B_2 = C^\infty M$.

Index theorem

DEFINITION: Let B_1, B_2 be vector bundles, amd $D: B_1 \longrightarrow B_2$ an elliptic operator. Its **index** $\operatorname{ind}(D) = \ker D - \ker D^*$.

REMARK: From elliptic regularity it follows that $\operatorname{ind} D$ **is equal to the index of the corresponding Fredholm map** $D: (B_1, L^2_p) \longrightarrow (B_2, L^2)$.

COROLLARY: Let D_t be a continuous family of elliptic operators. Then **the map** $t \mapsto \operatorname{ind}(D_t)$ **is constant.**

Proof: All D_t are Fredholm, and the index is locally constant on the space of Fredholm operators. ■

REMARK: The symbol of a differential operator can be considered as a nowhere degenerate section of the bundle $\operatorname{Sym}^n TM$ for differential operators from $C^\infty M$ to $C^\infty M$, or of $\operatorname{Sym}^n TM \otimes \operatorname{End}(B)$ for differential operators on the vector bundle B. We have just proven that the index **is the topological invariant of the section**. Atiyah-Singer index theorem **expresses this number through the appropriate K-theoretic invariants** (Chern classes of a certain explicitly defined element in K-theory associated with the symbol).

Index theorem for elliptic operators on $C^\infty M$

THEOREM: Any elliptic operator on $C^\infty M$, $\dim M > 1$, has index 0.

Proof. Step 1: A differential operator D from $C^\infty M$ to $C^\infty M$ is elliptic iff its index is strictly positive or strictly negative on T^*M, otherwise the symbol will vanish somewhere on $T^*M\backslash 0$ by continuity. **Then the order of D is even, indeed, all homogeneous odd degree polynomials define odd functions.**

Step 2: For D elliptic, $\operatorname{symb}(D)$ is positive or negative everywhere; fixing the sign, we may assume it is positive. Clearly, the set of such symbols is convex, hence connected. This implies that **all elliptic operators on $C^\infty M$ have the same index.**

Step 3: Clearly, a self-adjoint operator on a bundle B has index 0; indeed, $\dim\ker A = \dim\operatorname{coker} A^*$ for any linear map. **It remains only to produce a self-adjoint operator on $C^\infty M$ for each even rank.** Let $\Delta = d^*d$ be the Laplacian; clearly, it is self-adjoint, and its symbol is g^{-1}, where $g \in \operatorname{Sym}^2 T^*M$ is the Riemannian form. Therefore, Δ and Δ^k are elliptic and self-adjoint. ■

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Emmy Noether und Fritz Noether, 1933