Complex surfaces

lecture 10: Gauduchon metrics

Misha Verbitsky

IMPA, sala 236

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Hermitian structures and positive forms

Let (V, I) be a vector space with complex structure, $\dim_{\mathbb{R}} V = 2n$ and g a Hermitian metric, that is, I-invariant scalar product. Then $\omega(\cdot,\cdot) := g(I\cdot,\cdot)$ is skew-symmetric: $g(Ix, y) = g(I^2x, Iy) = -g(x, Iy) = g(Iy, x)$. This 2-form is called **the Hermitian form**.

Theorem 1: Let ω be a Hermitian form, and ω_1 be any (1,1)-form. **Then there exist orthonormal coordinates $x_1, ..., x_n, y_1, ..., y_n, y_i = I(x_i)$ such that $\omega = \sum_i x_i \wedge y_i$ and $\omega_1 = \sum_i \alpha_i x_i \wedge y_i$.** Moreover, ω_i has signature (p, q) if and only if p numbers α_i are positive, and q are negative.

Proof: See, for example, `http://verbit.ru/ULB/Alg-2016/slides-alg-10.pdf`. ■

DEFINITION: A (1,1)-form $\eta \in \Lambda^{1,1}_{\mathbb{R}}(V^*)$ is called **positive** if $\eta(x, Ix) \geqslant 0$ for all x, and **strictly positive** if the inequality is strict for all $x \neq 0$.

REMARK: From the previous theorem, we obtain that **a positive form η can be represented as $\sum_i \alpha_i x_i \wedge y_i$, $\alpha_i \geqslant 0$** in some orthonormal basis.

Positive $(n-1,n-1)$-forms

REMARK: The pairing $\Lambda^{1,1}(V)\otimes\Lambda^{n-1,n-1}(V)\longrightarrow\Lambda^{n,n}(V)=\operatorname{Vol}(V)$ is clearly non-degenerate. Choosing a volume form Vol, we might identify $\Lambda^{n-1,n-1}(V^*)$ and $\Lambda^{1,1}(V)$, using the formula $\Psi(z)=i_z\operatorname{Vol}$, where i_z is a contraction of the volume form with a bivector.

CLAIM: Let $P\in\Lambda^{n-1,n-1}(V^*)$ be a real $(n-1,n-1)$-form. **Then the following are equivalent:**

(i) Let $z\in\Lambda^{1,1}(V)$ be a bivector such that $i_z\operatorname{Vol}=P$. Then z is positive as a (1,1)-form on V^*.

(ii) For any Hermitian metric on V there exists an orthonormal basis $x_1,...,x_n,y_1,...,y_n\in V^*$ such that $I(x_i)=y_i$ and $z=\sum_{k=1}^n\alpha_k u_k$, where $\alpha_k\geqslant 0$ and $u_k=x_1\wedge y_1\wedge...\wedge x_{k-1}\wedge y_{k-1}\wedge x_{k+1}\wedge y_{k+1}\wedge...\wedge x_n\wedge y_n$ is the monomial obtained as the exterior product of all $x_i\wedge y_i$ except $x_k\wedge y_k$.

(iii) for any real 1-form $\lambda\in\Lambda^1(V^*)$, the volume form $\lambda\wedge I(\lambda)\wedge P$ is non-negative.

Proof. Step 1: We prove (i) $\Rightarrow$ (ii). Choose a Hermitian form on V, and, therefore, on V^*. Assume that Vol is the corresponding Riemannian volume form. Then there exists an orthonormal basis $x_1, ..., x_n, y_1, ..., y_n$, $I(x_i) = y_i$ in V^* such that $z = \sum_k \alpha_k x_k^* \wedge y_k^*$ and $\alpha_i \geqslant 0$. Then $\eta = \sum_{k=1}^n \alpha_k u_k$, because $\mathsf{Vol} = x_1 \wedge y_1 \wedge ... \wedge x_n \wedge y_n$ and $i_{\alpha_k x_k^* \wedge y_k^*}(\mathsf{Vol}) = u_k$.

Step 2: We prove (ii) $\Rightarrow$ (iii). Clearly, it suffices to show that $\lambda \wedge I(\lambda) \wedge u_i \geqslant 0$. Let $\Pi_i : V^* \longrightarrow \langle x_i, I(x_i)\rangle$ be the orthogonal projection. Then $\lambda \wedge I(\lambda) \wedge u_i = \Pi(\lambda) \wedge I(\Pi(\lambda)) \wedge u_i$, and this form is proportional to Vol with coefficient which is equal to $\frac{\Pi(\lambda)\wedge I(\Pi(\lambda))}{x_i \wedge y_i}$. This coefficient is $\geqslant 0$ because the volume form $a \wedge I(a)$ on (R^2, I) is $\geqslant 0$ for all a.

Step 3: It remains to prove (iii) $\Rightarrow$ (i). Consider the 2-form taking $a, b \in V^*$ to $\frac{P \wedge a \wedge b}{\mathsf{Vol}}$. Since $P = i_z \mathsf{Vol}$, this form is equal to $a, b \mapsto \frac{i_z(\mathsf{Vol}) \wedge a \wedge b}{\mathsf{Vol}}$, which is the same as $a, b \mapsto z(a, b)$, hence P satisfies (iii) whenever z is positive. ■

DEFINITION: An $(n-1, n-1)$-form $P \in \Lambda^{n-1,n-1}(V^*)$ is called **positive** if one of these conditions hold, and **strictly positive** if the inequalities in (i)-(iii) are strict (in other words, **the strictly positive forms lie in the interior of the cone of positive forms).**

Positive $(n-1,n-1)$-forms as products of (1,1)-forms

CLAIM: An $(n-1)$-th power of a positive (1,1)-form **is positive.** Conversely, any strictly positive $(n-1,n-1)$-form P is an $(n-1)$-th power of a positive (1,1)-form ω, which is uniquely determined by P.

Proof. Step 1: A positive form is $\omega := \sum_k \alpha_k x_k \wedge y_k$, and $\omega^{n-1} = \sum_l \beta_l u_l$, where $\beta_l = \prod_{i\neq l} \alpha_i$. **This proves (i).**

Step 2: Any positive $(n-1,n-1)$-form $P = \sum_k \beta_k u_k$ is a sum of monomials $u_k = x_1 \wedge y_1 \wedge ... \wedge x_{k-1} \wedge y_{k-1} \wedge x_{k+1} \wedge y_{k+1} \wedge ... \wedge x_n \wedge y_n$. Strict positivity means that all β_i are positive. If $P = \omega^{n-1}$, where $\omega = \sum_k \alpha_k x_k \wedge y_k$, we have $\beta_l = \prod_{i\neq l} \alpha_i$. This gives $\prod_i \alpha_l = \sqrt[n-1]{\prod_i \beta_i}$, hence $\alpha_k = \beta_k^{-1} \sqrt[n-1]{\prod_i \beta_i}$. **This proves (iii).**

Step 3: Conversely, let $\omega := \sum_k \alpha_k x_k \wedge y_k$, where $\alpha_k = \beta_k^{-1} \sqrt[n-1]{\prod_i \beta_i}$ Then $omega^{n-1} = \sum_l \beta_l u_l = P$ by the same formula. ■

REMARK: We have just proved that the map $\omega \mapsto \omega^{n-1}$ **defines a homeomorphism** from the cone of strictly positive $(1,1)$-forms to the cone of strictly positive $(n-1,n-1)$-forms.

Harnack inequality

DEFINITION: Let $U \subset V$ be two sets in a topological space. We say that **U is relatively compact in V**, denoted $U \Subset V$ if the closure of U is compact and belongs to V.

THEOREM: Let $\Omega \Subset \Omega_1$ be open sets, and $L: C^\infty\Omega_1 \longrightarrow C^\infty\Omega_1$ an elliptic operator. Then there exists a number $C > 1$ depending on L, Ω and Ω_1 such that **for any function $u \in \ker L$ such that $u \geqslant 0$, we have $\sup_\Omega u \leqslant C \inf_\Omega u$.**

Proof: Gilbarg-Trudinger, Corollary 8.21. ∎

COROLLARY: In assumptions of previous theorem, **$u > 0$ on Ω unless $u = 0$ identically.**

Gauduchon metrics

DEFINITION: Let (M, I, ω) be a Hermitian complex n manifold. The Hermitian metric ω is called **Gauduchon** if $dd^c(\omega^{n-1}) = 0$.

The following theorem is the most fundamental and important result in theory of complex manifolds; this is the only known result which works for all compact complex manifolds, and not for some special classes.

THEOREM: (Paul Gauduchon, 1977) Let (M, I, ω) be a compact complex Hermitian n-manifold. Then **there exists a unique (up to a constant multiplier) positive smooth function ψ such that $\psi\omega$ is Gauduchon.**

Paul Gauduchon (born March 22, 1945)

Gauduchon theorem

Proof. Step 1: It would suffice find a function $\varphi > 0$ such that $\partial\bar\partial(\varphi\omega^{n-1}) = 0$. Then $\varphi = \psi^{1/(n-1)}$.

Step 2: Consider a differential operator $L(\varphi) = \frac{\partial\bar\partial(\varphi\omega^{n-1})}{\omega^n}$ It is easy to see that **L is elliptic with the same symbol as the Laplacian (prove it).**

Step 3: Consider an L^2-metric on the space $C^\infty M$, defined by $(x,y)_{L^2} = \int_M xy\omega^n$. Its adjoint operator can be written explicitly, as follows. Let $\alpha \in C^\infty M$. Stokes' formula gives

$$(L(\varphi),\alpha)_{L^2} = \int_M L(\varphi)\alpha\omega^n = \int_M \partial\bar\partial(\varphi\omega^{n-1})\alpha = \int_M \varphi\omega^{n-1}\wedge\partial\bar\partial\alpha = (\varphi, L^*(\alpha))_{L^2},$$

hence $L^*\alpha = \frac{\omega^{n-1}\wedge\partial\bar\partial\alpha}{\omega^n}$.

Step 4: Since L^* vanishes on constants, its kernel is 1-dimensional by the strong maximum principle. Index formula for elliptic operators on functions (Lecture 9) implies that $\operatorname{coker} L^*$ is also 1-dimensional. Then $\ker L$ is 1-dimensional as well. This implies that a non-zero solution of $L(f) = 0$ exists, and is unique up to a constant. **To prove Gauduchon's theorem, it remains only to show that any non-zero function $u \in \ker L$ is everywhere positive, or everywhere negative.**

Gauduchon theorem (2)

Step 5: Let $f = L^*(h)$. In any neighbourhood $U \subset M$ of a maximum of h, there exist a point $x \in U$ such that $f(u) < 0$, because otherwise h is a solution of $L^*(h) \geqslant 0$, and such solution cannot have a maximum by the strong maximum principle. This implies that **any $f \in \operatorname{im} L^*$ is strictly negative somewhere on M.**

Step 6: If $\ker L$ contains a function u which is strictly positive and strictly negative somewhere on M, we will construct a function $\alpha \in C^\infty M$, which is positive everywhere and satisfies $(u, \alpha)_{L^2} = 0$. Since $\operatorname{im} L^* = (\ker L)^\perp$, this implies that $\alpha \in \operatorname{im} L^*$. This is impossible by Step 5: any $\alpha \in \operatorname{im} L^*$ is strictly negative somewhere on M. Therefore, **$u \geqslant 0$ or $u \leqslant 0$ everywhere on M.** This implies that $u > 0$ or $u < 0$ by Harnack inequality. ■