Complex surfaces

lecture 11: Bott-Chern cohomology and the defect of a complex surface

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Bott-Chern cohomology

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{BC}(M)$ the space of closed (p,q)-forms modulo $dd^c(\Lambda^{p-1,q-1}(M))$. Then $H^{p,q}_{BC}(M)$ is called **the Bott-Chern cohomology** of M.

REMARK: There are natural (and functorial) **maps from the Bott-Chern cohomology to the Dolbeault cohomology $H^*(\Lambda^{*,*}(M), \overline{\partial})$ and to the de Rham cohomology,** but no morphisms between de Rham and Dolbeault cohomology.

REMARK: However, **there is no multiplicative structure on the Bott-Chern cohomology.**

THEOREM: Let M be a compact complex manifold. **Then $H^{p,q}_{BC}(M)$ is finite-dimensional.**

Proof: Later today.

Elliptic complexes

DEFINITION: Let F, G, H be vector bundles, and $F \xrightarrow{D} G \xrightarrow{D} H$ a complex of differential operators (that is, $D^2 = 0$). It is called **elliptic complex** if its symbols $F \xrightarrow{\sigma(D)} G \xrightarrow{\sigma(D)} H$ give an exact sequence at each non-zero $\xi \in T^*M$.

DEFINITION: Let A, B, C be topological vector spaces and $A \xrightarrow{D} B \xrightarrow{D} C$ a complex of continuous maps. It is called **Fredholm complex** if $\operatorname{im} D$ is closed, and $\frac{\ker D}{\operatorname{im} D}$ is finite-dimensional.

THEOREM: Let $F \xrightarrow{D_1} G \xrightarrow{D_2} H$ be an elliptic complex of differential operators, with D_1 of order d_1 and D_2 of order d_2. **Then the complex $L^2_p(F) \xrightarrow{D_1} L^2_{p-d_1}(G) \xrightarrow{D_2} L^2_{p-d_1-d_2}(H)$ is Fredholm.**

COROLLARY: **Cohomology of any elliptic complex are finite-dimensional.**

Bott-Chern cohomology groups are finite-dimensional

Now we can prove

THEOREM: Let M be a compact complex manifold. **Then $H^{p,q}_{BC}(M)$ is finite-dimensional.**

Proof: It would suffice to show that the complex

$$\Lambda^{p-1,q-1}(M) \xrightarrow{dd^c} \Lambda^{p,q}(M) \xrightarrow{\partial+\overline{\partial}} \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M)$$

is elliptic. **At $\xi \in T^*M$, symbol of dd^c is equal to the multiplication of a form by $\xi \wedge I(\xi)$, the symbol of ∂ is multiplication by $\xi^{1,0}$ and the symbol of $\overline{\partial}$ is multiplication by $\xi^{0,1}$.** Therefore, $\ker\sigma(\partial) = \operatorname{im}\sigma(\partial)$, $\ker\sigma(\overline{\partial}) = \operatorname{im}\sigma(\overline{\partial})$ (this proves finite-dimensionality of Dolbeault cohomology), and $\ker\sigma(\overline{\partial}) \cap \ker\sigma(\partial) = \operatorname{im}\sigma(\partial\overline{\partial})$. ∎

dd^c-lemma

THEOREM: Let η be an exact (p,q)-form on a compact Kähler manifold, **Then $\eta \in \operatorname{im} dd^c = \operatorname{im} \partial\bar\partial$.**

Proof. Step 1: Notice immediately that η **is closed and orthogonal to the kernel of Δ**, hence its cohomology class vanishes. Indeed, $\ker \Delta$ is orthogonal to the image of d. Since η is exact, it lies in the image of Δ. Operator $G_\Delta := \Delta^{-1}$ is defined on $\operatorname{im}\Delta = \ker \Delta^\perp$ and commutes with d, d^c.

Step 2: Then $\eta = d\alpha$, where $\alpha := G_\Delta d^* \eta$. Since G_Δ and d^* commute with d^c, the form α is d^c-closed; since it belongs to $\operatorname{im}\Delta = \operatorname{im} G_\Delta$, it is d^c-exact, $\alpha = d^c\beta$ which gives $\eta = dd^c\beta$. ■

REMARK: Let M be a manifold which satisfies the conclusion of dd^c-lemma. **Then the natural map $H^*_{BC}(M) \longrightarrow H^*(M)$ is injective**. The converse is also true: if $H^{*,*}_{BC}(M) \longrightarrow H^*(M)$ is injective, any d-exact (p,q)-form is dd^c-exact, hence M satisfies the dd^c-lemma.

Intersection form on $\operatorname{Re}\Lambda^{1,1}_{\text{prim}}(V)$

Lemma 1: Let (V, I, g) be a 4-dimensional space equipped with a complex structure operator $I \in \operatorname{End}(V)$, $I^2 = -\operatorname{Id}$, $\omega \in \Lambda^{1,1}(V)$ a Hermitian form, and $\Lambda^{1,1}_{\text{prim}}(V) \subset \Lambda^{1,1}(V, I) \subset \Lambda^4(V)$ be the space of (1,1)-forms α such that $\alpha \wedge \omega = 0$. **Then for any non-zero $\alpha \in W$, one has $\frac{\alpha\wedge\alpha}{\text{Vol}} < 0$.**

Proof. Step 1: Consider the Hodge star operator $*: \Lambda^2(V) \longrightarrow \Lambda^2(V)$. Clearly, $*^2 = \operatorname{Id}$, hence all eigenvalues of $*$ are ± 1. If we invert the orientation, $*$ becomes $-*$; this implies that $*$ is conjugated to $-*$, hence the multiplicity of 1 and -1 is equal 3. **Denote the corresponding eigenspaces as $\Lambda^2 V = \Lambda^+ V \oplus \Lambda^- V$.** This decomposition is clearly orthogonal with respect to the pairing $\alpha, \beta \longrightarrow \frac{\alpha\wedge\beta}{\text{Vol}}$.

Step 2: Consider a quaternion action on V compatible with the scalar product g. Three symplectic forms $\omega_I, \omega_J, \omega_K$ are pairwise orthogonal, square to 0, hence generate $\Lambda^+ V$. However, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is of type (2,0) on (V, I). **Therefore, $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle = \Lambda^+ V$.**

Step 3: The space $\Lambda^{1,1}_{\text{prim}}(V)$ is 3-dimensional and orthogonal to the 3-dimensional space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle$. The space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle$ is equal to $\Lambda^+ V$, as follows from Step 2. **Then $\Lambda^{1,1}_{\text{prim}}(V) = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle^\perp = \Lambda^- V$.** ■

DEFINITION: A (1,1)-form on a complex Hermitian surface is **primitive** if it is orthogonal to ω pointwise. **Primitive forms satisfy $\|\eta\|^2_{L^2} = -\int_M \eta\wedge\eta$.**

Bott-Chern cohomology of a surface

THEOREM: Let M be a compact surface. **Then the kernel of the natural map $P: H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is at most 1-dimensional.**

Proof. Step 1: Let ω be a Gauduchon metric on M. Consider the differential operator $D: f \mapsto dd^c(f) \wedge \omega$ mapping functions to 4-forms. Clearly, D is elliptic and its index is the same as the index of the Laplacian: $\operatorname{ind} D = \operatorname{ind} \Delta = 0$, hence $\dim \ker D = \dim \operatorname{coker} D$. The Hopf maximum principle implies that $\ker D$ only contains constants, hence by index theorem $\operatorname{coker} D$ is 1-dimensional. However, $\int_M D(f) = \int_M dd^c(f) \wedge \omega = \int_M f dd^c \omega = 0$. This implies that **a 4-form κ belongs to $\operatorname{im} D$ if and only if $\int_M \kappa = 0$.**

Step 2: Let α be a closed $(1,1)$-form. Define **the degree** $\deg_\omega \alpha := \int_M \omega \wedge \alpha$. Since $\int_M dd^c f \wedge \omega = 0$, this defines a map $\deg_\omega: H^{1,1}_{BC}(M, \mathbb{R}) \longrightarrow \mathbb{R}$. Given a closed $(1,1)$-form α of degree 0, the form $\alpha' := \alpha - dd^c(D^{-1}(\alpha \wedge \omega))$ satisfies $\alpha' \wedge \omega = 0$, in other words, it is an ω-primitive (1,1)-form. For ω-primitive forms, one has $\alpha' \wedge \alpha' = -|\alpha'|^2 \omega \wedge \omega$, giving $\int_M \alpha' \wedge \alpha' = -\|\alpha'\|^2_\omega$ which is impossible when α' is a non-zero class in $\ker P$, because then α' is exact. Therefore, **any vector of zero degree in $\ker P \subset H^{1,1}_{BC}(M, \mathbb{R})$ vanishes.** This implies that any two vectors in $\ker P$ are proportional. ■

Defect of a complex surface

COROLLARY: Let M be a complex surface and η be a non-zero vector in $\ker P$, where $P: H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is the natural map morphism. **Then $\int \eta \wedge \omega > 0$ for all Gauduchon forms ω, or $\int \eta \wedge \omega < 0$ for all ω.**

Proof: Consider two Gauduchon forms ω_1, ω_2 such that $\int \eta \wedge_1 \omega < 0$ and $\int \eta \wedge_2 \omega < 0$. **Then there is a Gauduchon form $\omega := a\omega_1 + b\omega_2$, $a, b > 0$ such that $\int \eta \wedge \omega = 0$, giving a contradiction.** ■

REMARK: A Bott-Chern class which satisfies $\int \eta \wedge \omega > 0$ for all Gauduchon forms **is represented by a positive current** (A. Lamari).

DEFINITION: The number $\dim \ker P$ is called **the defect** of a surface, denoted $\delta(M)$; by the previous theorem it can be 1 or 0. In the course of the proof of Lamari's theorem, we will show that **the surface is Kähler if and only if $\delta(M) = 1$.**