Complex surfaces

lecture 12: First cohomology of a complex surface

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Intersection form on $\operatorname{Re}\Lambda^{1,1}_{\text{prim}}(V)$ (reminder)

Lemma 1: Let (V, I, g) be a 4-dimensional space equipped with a complex structure operator $I \in \operatorname{End}(V)$, $I^2 = -\operatorname{Id}$, $\omega \in \Lambda^{1,1}(V)$ a Hermitian form, and $\Lambda^{1,1}_{\text{prim}}(V) \subset \Lambda^{1,1}(V, I) \subset \Lambda^4(V)$ be the space of (1,1)-forms α such that $\alpha \wedge \omega = 0$. **Then for any non-zero $\alpha \in W$, one has $\frac{\alpha\wedge\alpha}{\text{Vol}} < 0$.**
Proof. Step 1: Consider the Hodge star operator $* : \Lambda^2(V) \longrightarrow \Lambda^2(V)$. Clearly, $*^2 = \operatorname{Id}$, hence all eigenvalues of $*$ are ± 1. If we invert the orientation, $*$ becomes $-*$; this implies that $*$ is conjugated to $-*$, hence the multiplicity of 1 and -1 is equal 3. **Denote the corresponding eigenspaces as $\Lambda^2 V = \Lambda^+ V \oplus \Lambda^- V$.** This decomposition is clearly orthogonal with respect to the pairing $\alpha, \beta \longrightarrow \frac{\alpha\wedge\beta}{\text{Vol}}$.

Step 2: Consider a quaternion action on V compatible with the scalar product g. Three symplectic forms $\omega_I, \omega_J, \omega_K$ are pairwise orthogonal, square to 0, hence generate $\Lambda^+ V$. However, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is of type (2,0) on (V, I). **Therefore, $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle = \Lambda^+ V$.**

Step 3: The space $\Lambda^{1,1}_{\text{prim}}(V)$ is 3-dimensional and orthogonal to the 3-dimensional space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle$. The space $\langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle$ is equal to $\Lambda^+ V$, as follows from Step 2. **Then $\Lambda^{1,1}_{\text{prim}}(V) = \langle \operatorname{Re}\Omega, \operatorname{Im}\Omega, \omega\rangle^\perp = \Lambda^- V$.** ■

DEFINITION: A (1,1)-form on a complex Hermitian surface is **primitive** if it is orthogonal to ω pointwise. **Primitive forms satisfy $\|\eta\|^2_{L^2} = -\int_M \eta \wedge \eta$.**

Bott-Chern cohomology (reminder)

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{BC}(M)$ the space of closed (p,q)-forms modulo $dd^c(\Lambda^{p-1,q-1}(M))$. Then $H^{p,q}_{BC}(M)$ is called **the Bott-Chern cohomology** of M.

REMARK: There are natural (and functorial) **maps from the Bott-Chern cohomology to the Dolbeault cohomology $H^*(\Lambda^{*,*}(M),\bar\partial)$ and to the de Rham cohomology,** but no morphisms between de Rham and Dolbeault cohomology.

REMARK: However, **there is no multiplicative structure on the Bott-Chern cohomology.**

THEOREM: Let M be a compact complex manifold. **Then $H^{p,q}_{BC}(M)$ is finite-dimensional.**

Proof: Lecture 11.

Complex surfaces, Bott-Chern cohomology and primitive forms (reminder)

Theorem 1: Let M be a compact surface. **Then the kernel of the natural map $P: H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is at most 1-dimensional.**

Proof. Step 1: Let ω be a Gauduchon metric on M. Consider the differential operator $D: f \mapsto dd^c(f) \wedge \omega$ mapping functions to 4-forms. Clearly, D is elliptic and its index is the same as the index of the Laplacian: $\operatorname{ind} D = \operatorname{ind} \Delta = 0$, hence $\dim\ker D = \dim\operatorname{coker} D$. The Hopf maximum principle implies that $\ker D$ only contains constants, hence by index theorem $\operatorname{coker} D$ is 1-dimensional. However, $\int_M D(f) = \int_M dd^c(f) \wedge \omega = \int_M f dd^c\omega = 0$. This implies that **a 4-form κ belongs to $\operatorname{im} D$ if and only if $\int_M \kappa = 0$.**

Step 2: Let α be a closed $(1,1)$-form. Define **the degree** $\deg_\omega \alpha := \int_M \omega \wedge \alpha$. Since $\int_M dd^c f \wedge \omega = 0$, this defines a map $\deg_\omega: H^{1,1}_{BC}(M,\mathbb{R}) \longrightarrow \mathbb{R}$. Given a closed $(1,1)$-form α of degree 0, the form $\alpha' := \alpha - dd^c(D^{-1}(\alpha\wedge\omega))$ satisfies $\alpha' \wedge \omega = 0$, in other words, it is an ω-primitive (1,1)-form. For ω-primitive forms, one has $\alpha' \wedge \alpha' = -|\alpha'|^2 \omega\wedge\omega$, giving $\int_M \alpha' \wedge \alpha' = -\|\alpha'\|^2_\omega$ which is impossible when α' is a non-zero class in $\ker P$, because then α' is exact. Therefore, **any vector of zero degree in $\ker P \subset H^{1,1}_{BC}(M,\mathbb{R})$ vanishes.** This implies that any two vectors in $\ker P$ are proportional. ∎

Defect of a complex surface (reminder)

COROLLARY: Let $\eta \in H^{1,1}_{BC}(M)$ be a non-zero d-exact class. **Then $\int \eta \wedge \omega \neq 0$.**

Proof: Follows from Step 2 of the previous theorem

COROLLARY: Let M be a complex surface and η be a non-zero vector in $\ker P$, where $P: \ H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is the natural map morphism. **Then $\int \eta \wedge \omega > 0$ for all Gauduchon forms ω, or $\int \eta \wedge \omega < 0$ for all ω.**

Proof: Follows from the above corollary.

DEFINITION: The number $\dim \ker P$ is called **the defect** of a surface, denoted $\delta(M)$; by the previous theorem it can be 1 or 0. In the course of the proof of Lamari's theorem, we will show that **the surface is Kähler if and only if $\delta(M) = 1$.**

Intersection form on $H^{1,1}_{BC}(M)$

PROPOSITION: Let M be a compact surface with $\delta(M) > 0$. **Then the intersection form on the image of $H^{1,1}_{BC}(M,\mathbb{R})$ in $H^2(M,\mathbb{R})$ is negative definite.**

Proof: Fix a Gauduchon metric ω on M. Consider the degree functional $\deg_\omega : H^{1,1}_{BC}(M,\mathbb{R}) \longrightarrow \mathbb{R}$ (Lecture 11) taking $\alpha \in H^{1,1}_{BC}(M,\mathbb{R})$ to $\int_M \alpha\wedge\omega$. Then $\deg_\omega(\Theta) \neq 0$ for any non-zero d-exact class $\Theta \in \ker P : H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ (Lecture 11). Therefore, any class in $\frac{H^{1,1}_{BC}(M,\mathbb{R})}{\ker P}$ can be represented by a closed (1,1)-form α with $\deg_\omega \alpha = 0$. Acting as in the proof of Theorem 1, we find $f \in C^\infty(M)$ such that $\alpha - dd^c f$ is primitive. Replacing α by $\alpha' := \alpha - dd^c f$, we obtain $\int_M \alpha' \wedge \alpha' = -\|\alpha'\|^2_\omega < 0$. ■

Holomorphic 1-forms on a surface

LEMMA: All holomorphic 1-forms on a compact complex surface are closed.

Proof: Let $\alpha \in \Lambda^{1,0}(M)$ be a holomorphic 1-form. Then $\bar\partial\alpha = 0$, because it is holomorphic, and by the same reason $d\alpha$ is a holomorphic, exact (2,0)-form. Then $d\alpha \wedge d\overline{\alpha}$ is a positive (2,2)-form, giving $0 = \int_M d\alpha \wedge d\overline{\alpha} = \|d\alpha\|^2$. Then $d\alpha = 0$, and α is closed. ■

Claim 1: Let M be a complex surface. Denote the space of holomorphic 1-forms on M by $\mathcal{H}^{1,0}(M)$, and let $\overline{\mathcal{H}^{1,0}(M)}$ be its complex conjugate. By the previous lemma, all elements of $\mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$ are closed. This defines a map $\mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H^1(M, \mathbb{C})$. **We claim that this map is injective.**

Proof: Let α, β be holomorphic forms such that $\alpha + \overline{\beta}$ is exact, $\alpha + \overline{\beta} = df$. Then $dd^c f = 0$, hence $f = const$ by maximum principle. Indeed, $f \mapsto \frac{dd^c f \wedge \omega}{\omega^2}$ is an elliptic operator, vanishing on constants, hence all dd^c-closed functions are constant. ■

$H^{0,1}_{\bar\partial}(M)$ for a complex surface

Claim 2: Consider the natural map $R: \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H^{0,1}_{\bar\partial}(M)$. **Then R is injective.**

Proof: For any α in its kernel, $\alpha = \bar\partial u$, but α is ∂-closed, hence $\partial\bar\partial u = 0$, implying $u = const$ by maximum principle. Therefore, R is injective. ■

Claim 3: Assume that $P: H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is injective, that is, $\delta(M) = 0$. **Then $R: \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H^{0,1}_{\bar\partial}(M)$ is surjective.**

Proof: If R is not surjective, there is a class represented by a $\bar\partial$-closed (0,1)-form α, but not by a closed (0,1)-form. Then $\partial(\alpha + \bar\partial\varphi) \neq 0$ for any function φ on M, which implies that $d\alpha$ generates $\ker P$. ■

Proposition 4: The following sequence is exact:

$$0 \longrightarrow \overline{\mathcal{H}^{1,0}(M)} \xrightarrow{R} H^{0,1}_{\bar\partial}(M) \xrightarrow{\partial} H^{1,1}_{BC}(M,\mathbb{R}) \xrightarrow{P} H^2(M)$$

In particular, P is injective if and only if R is surjective.

Proof: Exactness in the $H^{1,1}_{BC}(M,\mathbb{R})$-term follows from Claim 3. Exactness in $H^{0,1}_{\bar\partial}(M)$-term follows from the definition. Exactness in the first term follows from Claim 2. ■

$H^1(M)$ for a complex surface with $\delta(M)=0$

PROPOSITION: Let M be a compact complex manifold. **Then the map $H^1(M,\mathbb{R}) \xrightarrow{\tau} H^{0,1}_{\bar\partial}(M)$, taking a closed form η to $[\eta^{0,1}]$, is injective.**

Proof: Let η be a real form such that $\eta^{0,1}$ is $\bar\partial$-exact, $\eta^{0,1} = \bar\partial\varphi$, where $\varphi = a + \sqrt{-1}\, b$, where a, b are real functions. Then $\eta = 2\operatorname{Re}\bar\partial\varphi = (da - d^c b)$. We obtain that η is cohomologous to a form $d^c b$ which is d-closed and d^c-closed. **This gives $dd^c b = 0$, hence b is constant, by maximum principle, and $\eta = da$ is exact.** ■

CLAIM: Let M be a complex surface, such that $H^{1,1}_{BC}(M,\mathbb{R}) \xrightarrow{P} H^2(M)$ is injective, that is, $\delta(M)=0$. **Then $H^1(M,\mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$.**

Proof: If $\delta(M)=0$, then $R: \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H^{0,1}_{\bar\partial}(M)$ is an isomorphism by Proposition 4. Since all elements of $\overline{\mathcal{H}^{1,0}(M)}$ are closed, the natural map $\tau: H^1(M,\mathbb{R}) \longrightarrow H^{0,1}_{\bar\partial}(M)$ is surjective. It is injective by the previous proposition. Passing to its complexification, we obtain $H^1(M,\mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$. ■

$H^1(M)$ for a complex surface with $\delta(M)=1$

Proposition 5: Let M be a complex surface such that $H^{1,1}_{BC}(M,\mathbb{R}) \xrightarrow{P} H^2(M)$ has nonzero kernel, that is, $\delta(M)=1$. **Then $\ker P$ can be generated by a class $d^c[\theta]$, where $\theta\in H^1(M,\mathbb{R})$, and $H^1(M,\mathbb{C})=\mathcal{H}^{1,0}(M)\oplus\overline{\mathcal{H}^{1,0}(M)}\oplus\langle\theta\rangle$.**

Proof. Step 1: The generator u of $\ker P$ is a differential of a real 1-form α, which satisfies $\overline{\partial}\alpha^{0,1}=\partial\alpha^{1,0}=0$, hence $d\alpha^{0,1}=d\alpha^{1,0}=u$. Since u is a real form, the imaginary part of $d\alpha^{0,1}$ vanishes. Then $\theta:=I\alpha$ is closed, and $u=d^c\theta$, **hence the cohomology class $[\theta]\in H^1(M,\mathbb{R})$ is non-exact and linearly independent from $\mathcal{H}^{1,0}(M)\oplus\overline{\mathcal{H}^{1,0}(M)}$.**

Step 2: Using the exact sequence

$$0\longrightarrow\overline{\mathcal{H}^{1,0}(M)}\xrightarrow{R} H^{0,1}_{\overline{\partial}}(M)\xrightarrow{\partial} H^{1,1}_{BC}(M,\mathbb{R})\xrightarrow{P} H^2(M)$$

we obtain that $\langle\theta^{0,1}\rangle\oplus\overline{\mathcal{H}^{1,0}(M)}=H^{0,1}_{\overline{\partial}}(M)$.

Step 3: Since the natural map $\tau:\ H^1(M,\mathbb{R})\longrightarrow H^{0,1}_{\overline{\partial}}(M)$ is injective, and $H^{0,1}_{\overline{\partial}}(M)=\langle\theta^{0,1}\rangle\oplus\overline{\mathcal{H}^{1,0}(M)}$, we obtain that τ is surjective and $H^1(M,\mathbb{R})$ is generated by the real parts of $\overline{\mathcal{H}^{1,0}(M)}$ and $\langle\theta\rangle$. ■

COROLLARY: Let M be a complex surface. **Then $b_1(M)$ is odd when $\delta(M)=1$ and $b_1(M)$ is even when $\delta(M)=0$.**

Proof: When $\delta(M)=0$, we have $H^1(M,\mathbb{C})=\mathcal{H}^{1,0}(M)\oplus\overline{\mathcal{H}^{1,0}(M)}$, and when $\delta(M)=0$, we have $H^1(M,\mathbb{C})=\mathcal{H}^{1,0}(M)\oplus\overline{\mathcal{H}^{1,0}(M)}\oplus\langle\theta\rangle$. ■

$H^{0,1}_{\bar\partial}(M)$ for a complex surface

COROLLARY: Let M be a complex surface. **Then** $H^{0,1}_{\bar\partial}(M) = \overline{\mathcal{H}^{1,0}(M)}$ **when** $\delta(M) = 0$ **and** $H^{0,1}_{\bar\partial}(M) = \overline{\mathcal{H}^{1,0}(M)} \oplus \langle \theta^{0,1}\rangle$**,** where θ is the closed 1-form defined in Proposition 5, and $\theta^{0,1}$ the Dolbeault class of its (0,1)-part.

Proof: See Step 2 in the proof of Proposition 5. ■

Degeneration of the Hodge-de Rham-Frölicher spectral sequence

DEFINITION: Let M be a compact complex manifold. We say that **the Hodge-de Rham-Frölicher spectral sequence degenerates in the term $E_1^{p,q}$** if any class in $H^{p,q}_{\overline{\partial}}(M)$ can be represented by a $\overline{\partial}$-closed (p,q)-form α which satisfies $\partial\alpha \in \operatorname{im}\overline{\partial}$. It degenerates on the page $E_1^{*,*}$ if it degenerates for all terms $E_1^{p,q}$.

REMARK: The degeneration in E_1 is equivalent to $E_1^{p,q} = E_\infty^{p,q}$, where $E_*^{*,*}$ is the Hodge-de Rham-Frölicher spectral sequence. Indeed, $E_1^{p,q} = H^{p,q}_{\overline{\partial}}$, and the differential d_1 takes the Dolbeault class $\alpha \in E_1^{p,q} = H^{p,q}_{\overline{\partial}}(M)$ to an element of $H^{p+1,q}_{\overline{\partial}}(M)$ represented by $(d\alpha)^{p+1,q}$. The same argument also implies that all differentials $d_i, i \geqslant 1$ vanish. However, if all d_i vanish, the spectral sequence degenerates, giving $E_1^{p,q} = E_\infty^{p,q}$; conversely, this equivalence implies that all differentials vanish, and any class in $H^{p,q}_{\overline{\partial}}(M)$ can be represented by a form in $\ker d_1$, that is, by a closed form.

COROLLARY: Let M be a complex surface. **Then the Hodge-de Rham-Frölicher spectral sequence degenerates in $E_1^{0,1}$ and in $E_1^{1,0}$.**

Proof: Indeed, all forms in $\mathcal{H}^{1,0}(M) = E_1^{1,0}$ are closed, and $E_1^{0,1} = \overline{\mathcal{H}^{1,0}(M)} \oplus \langle\theta^{0,1}\rangle$; all forms in $\overline{\mathcal{H}^{1,0}(M)}$ are closed, and $\partial(\theta^{0,1}) = \overline{\partial}(\theta^{1,0})$. ■