

Complex surfaces

lecture 12: First cohomology of a complex surface

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February 3, 2024, 17:00

Intersection form on $\operatorname{Re} \Lambda_{\text{prim}}^{1,1}(V)$ (reminder)

Lemma 1: Let (V, I, g) be a 4-dimensional space equipped with a complex structure operator $I \in \operatorname{End}(V)$, $I^2 = -\operatorname{Id}$, $\omega \in \Lambda^{1,1}(V)$ a Hermitian form, and $\Lambda_{\text{prim}}^{1,1}(V) \subset \Lambda^{1,1}(V, I) \subset \Lambda^4(V)$ be the space of (1,1)-forms α such that $\alpha \wedge \omega = 0$. **Then for any non-zero $\alpha \in W$, one has $\frac{\alpha \wedge \alpha}{\operatorname{Vol}} < 0$.**

Proof. Step 1: Consider the Hodge star operator $*$: $\Lambda^2(V) \rightarrow \Lambda^2(V)$. Clearly, $*^2 = \operatorname{Id}$, hence all eigenvalues of $*$ are ± 1 . If we invert the orientation, $*$ becomes $-*$; this implies that $*$ is conjugated to $-*$, hence the multiplicity of 1 and -1 is equal 3. **Denote the corresponding eigenspaces as $\Lambda^2 V = \Lambda^+ V \oplus \Lambda^- V$.** This decomposition is clearly orthogonal with respect to the pairing $\alpha, \beta \rightarrow \frac{\alpha \wedge \beta}{\operatorname{Vol}}$.

Step 2: Consider a quaternion action on V compatible with the scalar product g . Three symplectic forms $\omega_I, \omega_J, \omega_K$ are pairwise orthogonal, square to 0, hence generate $\Lambda^+ V$. However, $\Omega := \omega_J + \sqrt{-1} \omega_K$ is of type (2,0) on (V, I) . **Therefore, $\langle \operatorname{Re} \Omega, \operatorname{Im} \Omega, \omega \rangle = \Lambda^+ V$.**

Step 3: The space $\Lambda_{\text{prim}}^{1,1}(V)$ is 3-dimensional and orthogonal to the 3-dimensional space $\langle \operatorname{Re} \Omega, \operatorname{Im} \Omega, \omega \rangle$. The space $\langle \operatorname{Re} \Omega, \operatorname{Im} \Omega, \omega \rangle$ is equal to $\Lambda^+ V$, as follows from Step 2. **Then $\Lambda_{\text{prim}}^{1,1}(V) = \langle \operatorname{Re} \Omega, \operatorname{Im} \Omega, \omega \rangle^\perp = \Lambda^- V$. ■**

DEFINITION: A (1,1)-form on a complex Hermitian surface is **primitive** if it is orthogonal to ω pointwise. **Primitive forms satisfy $\|\eta\|_{L^2}^2 = -\int_M \eta \wedge \eta$.**

Bott-Chern cohomology (reminder)

DEFINITION: Let M be a complex manifold, and $H_{BC}^{p,q}(M)$ the space of closed (p, q) -forms modulo $dd^c(\Lambda^{p-1, q-1}(M))$. Then $H_{BC}^{p,q}(M)$ is called **the Bott-Chern cohomology** of M .

REMARK: There are natural (and functorial) **maps from the Bott-Chern cohomology to the Dolbeault cohomology** $H^*(\Lambda^{*,*}(M), \bar{\partial})$ and to the **de Rham cohomology**, but no morphisms between de Rham and Dolbeault cohomology.

REMARK: However, **there is no multiplicative structure on the Bott-Chern cohomology**.

THEOREM: Let M be a compact complex manifold. **Then** $H_{BC}^{p,q}(M)$ **is finite-dimensional**.

Proof: Lecture 11.

Complex surfaces, Bott-Chern cohomology and primitive forms (reminder)

Theorem 1: Let M be a compact surface. **Then the kernel of the natural map $P : H_{BC}^{1,1}(M) \rightarrow H^2(M)$ is at most 1-dimensional.**

Proof. Step 1: Let ω be a Gauduchon metric on M . Consider the differential operator $D : f \mapsto dd^c(f) \wedge \omega$ mapping functions to 4-forms. Clearly, D is elliptic and its index is the same as the index of the Laplacian: $\text{ind } D = \text{ind } \Delta = 0$, hence $\dim \ker D = \dim \text{coker } D$. The Hopf maximum principle implies that $\ker D$ only contains constants, hence by index theorem $\text{coker } D$ is 1-dimensional. However, $\int_M D(f) = \int_M dd^c(f) \wedge \omega = \int_M f dd^c \omega = 0$. This implies that **a 4-form κ belongs to $\text{im } D$ if and only if $\int_M \kappa = 0$.**

Step 2: Let α be a closed $(1,1)$ -form. Define **the degree** $\deg_\omega \alpha := \int_M \omega \wedge \alpha$. Since $\int_M dd^c f \wedge \omega = 0$, this defines a map $\deg_\omega : H_{BC}^{1,1}(M, \mathbb{R}) \rightarrow \mathbb{R}$. Given a closed $(1,1)$ -form α of degree 0, the form $\alpha' := \alpha - dd^c(D^{-1}(\alpha \wedge \omega))$ satisfies $\alpha' \wedge \omega = 0$, in other words, it is an ω -primitive $(1,1)$ -form. For ω -primitive forms, one has $\alpha' \wedge \alpha' = -|\alpha'|^2 \omega \wedge \omega$, giving $\int_M \alpha' \wedge \alpha' = -\|\alpha'\|_\omega^2$ which is impossible when α' is a non-zero class in $\ker P$, because then α' is exact. Therefore, **any vector of zero degree in $\ker P \subset H_{BC}^{1,1}(M, \mathbb{R})$ vanishes.** This implies that any two vectors in $\ker P$ are proportional. ■

Defect of a complex surface (reminder)

COROLLARY: Let $\eta \in H_{BC}^{1,1}(M)$ be a non-zero d -exact class.

Then $\int \eta \wedge \omega \neq 0$.

Proof: Follows from Step 2 of the previous theorem

COROLLARY: Let M be a complex surface and η be a non-zero vector in $\ker P$, where $P : H_{BC}^{1,1}(M) \rightarrow H^2(M)$ is the natural map morphism. **Then** $\int \eta \wedge \omega > 0$ for all Gauduchon forms ω , or $\int \eta \wedge \omega < 0$ for all ω .

Proof: Follows from the above corollary.

DEFINITION: The number $\dim \ker P$ is called **the defect** of a surface, denoted $\delta(M)$; by the previous theorem it can be 1 or 0. In the course of the proof of Lamari's theorem, we will show that **the surface is Kähler if and only if $\delta(M) = 1$.**

Intersection form on $H_{BC}^{1,1}(M)$

PROPOSITION: Let M be a compact surface with $\delta(M) > 0$. **Then the intersection form on the image of $H_{BC}^{1,1}(M, \mathbb{R})$ in $H^2(M, \mathbb{R})$ is negative definite.**

Proof: Fix a Gauduchon metric ω on M . Consider the degree functional $\deg_\omega : H_{BC}^{1,1}(M, \mathbb{R}) \rightarrow \mathbb{R}$ (Lecture 11) taking $\alpha \in H_{BC}^{1,1}(M, \mathbb{R})$ to $\int_M \alpha \wedge \omega$. Then $\deg_\omega(\Theta) \neq 0$ for any non-zero d -exact class $\Theta \in \ker P : H_{BC}^{1,1}(M) \rightarrow H^2(M)$ (Lecture 11). Therefore, any class in $\frac{H_{BC}^{1,1}(M, \mathbb{R})}{\ker P}$ can be represented by a closed $(1,1)$ -form α with $\deg_\omega \alpha = 0$. Acting as in the proof of Theorem 1, we find $f \in C^\infty(M)$ such that $\alpha - dd^c f$ is primitive. Replacing α by $\alpha' := \alpha - dd^c f$, we obtain $\int_M \alpha' \wedge \alpha' = -\|\alpha'\|_\omega^2 < 0$. ■

Holomorphic 1-forms on a surface

LEMMA: All holomorphic 1-forms on a compact complex surface are closed.

Proof: Let $\alpha \in \Lambda^{1,0}(M)$ be a holomorphic 1-form. Then $\bar{\partial}\alpha = 0$, because it is holomorphic, and by the same reason $d\alpha$ is a holomorphic, exact (2,0)-form. Then $d\alpha \wedge d\bar{\alpha}$ is a positive (2,2)-form, giving $0 = \int_M d\alpha \wedge d\bar{\alpha} = \|d\alpha\|^2$. Then $d\alpha = 0$, and α is closed. ■

Claim 1: Let M be a complex surface. Denote the space of holomorphic 1-forms on M by $\mathcal{H}^{1,0}(M)$, and let $\overline{\mathcal{H}^{1,0}(M)}$ be its complex conjugate. By the previous lemma, all elements of $\mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$ are closed. This defines a map $\mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H^1(M, \mathbb{C})$. **We claim that this map is injective.**

Proof: Let α, β be holomorphic forms such that $\alpha + \bar{\beta}$ is exact, $\alpha + \bar{\beta} = df$. Then $dd^c f = 0$, hence $f = \text{const}$ by maximum principle. Indeed, $f \mapsto \frac{dd^c f \wedge \omega}{\omega^2}$ is an elliptic operator, vanishing on constants, hence all dd^c -closed functions are constant. ■

$H_{\bar{\partial}}^{0,1}(M)$ for a complex surface

Claim 2: Consider the natural map $R : \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H_{\bar{\partial}}^{0,1}(M)$. **Then R is injective.**

Proof: For any α in its kernel, $\alpha = \bar{\partial}u$, but α is ∂ -closed, hence $\partial\bar{\partial}u = 0$, implying $u = \text{const}$ by maximum principle. Therefore, R is injective. ■

Claim 3: Assume that $P : H_{BC}^{1,1}(M) \longrightarrow H^2(M)$ is injective, that is, $\delta(M) = 0$. **Then $R : \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H_{\bar{\partial}}^{0,1}(M)$ is surjective.**

Proof: If R is not surjective, there is a class represented by a $\bar{\partial}$ -closed $(0,1)$ -form α , but not by a closed $(0,1)$ -form. Then $\partial(\alpha + \bar{\partial}\varphi) \neq 0$ for any function φ on M , which implies that $d\alpha$ generates $\ker P$. ■

Proposition 4: The following sequence is exact:

$$0 \longrightarrow \overline{\mathcal{H}^{1,0}(M)} \xrightarrow{R} H_{\bar{\partial}}^{0,1}(M) \xrightarrow{\partial} H_{BC}^{1,1}(M, \mathbb{R}) \xrightarrow{P} H^2(M)$$

In particular, P is injective if and only if R is surjective.

Proof: Exactness in the $H_{BC}^{1,1}(M, \mathbb{R})$ -term follows from Claim 3. Exactness in $H_{\bar{\partial}}^{0,1}(M)$ -term follows from the definition. Exactness in the first term follows from Claim 2. ■

$H^1(M)$ for a complex surface with $\delta(M) = 0$

PROPOSITION: Let M be a compact complex manifold. **Then the map $H^1(M, \mathbb{R}) \xrightarrow{\tau} H_{\bar{\partial}}^{0,1}(M)$, taking a closed form η to $[\eta^{0,1}]$, is injective.**

Proof: Let η be a real form such that $\eta^{0,1}$ is $\bar{\partial}$ -exact, $\eta^{0,1} = \bar{\partial}\varphi$, where $\varphi = a + \sqrt{-1}b$, where a, b are real functions. Then $\eta = 2 \operatorname{Re} \bar{\partial}\varphi = (da - d^c b)$. We obtain that η is cohomologous to a form $d^c b$ which is d -closed and d^c -closed. **This gives $dd^c b = 0$, hence b is constant, by maximum principle, and $\eta = da$ is exact. ■**

CLAIM: Let M be a complex surface, such that $H_{BC}^{1,1}(M, \mathbb{R}) \xrightarrow{P} H^2(M)$ is injective, that is, $\delta(M) = 0$. **Then $H^1(M, \mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$.**

Proof: If $\delta(M) = 0$, then $R : \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H_{\bar{\partial}}^{0,1}(M)$ is an isomorphism by Proposition 4. Since all elements of $\overline{\mathcal{H}^{1,0}(M)}$ are closed, the natural map $\tau : H^1(M, \mathbb{R}) \longrightarrow H_{\bar{\partial}}^{0,1}(M)$ is surjective. It is injective by the previous proposition. Passing to its complexification, we obtain $H^1(M, \mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$. ■

$H^1(M)$ for a complex surface with $\delta(M) = 1$

Proposition 5: Let M be a complex surface such that $H_{BC}^{1,1}(M, \mathbb{R}) \xrightarrow{P} H^2(M)$ has nonzero kernel, that is, $\delta(M) = 1$. **Then $\ker P$ can be generated by a class $d^c[\theta]$, where $\theta \in H^1(M, \mathbb{R})$, and $H^1(M, \mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \oplus \langle \theta \rangle$.**

Proof. Step 1: The generator u of $\ker P$ is a differential of a real 1-form α , which satisfies $\bar{\partial}\alpha^{0,1} = \partial\alpha^{1,0} = 0$, hence $d\alpha^{0,1} = d\alpha^{1,0} = u$. Since u is a real form, the imaginary part of $d\alpha^{0,1}$ vanishes. Then $\theta := I\alpha$ is closed, and $u = d^c\theta$, **hence the cohomology class $[\theta] \in H^1(M, \mathbb{R})$ is non-exact and linearly independent from $\mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$.**

Step 2: Using the exact sequence

$$0 \longrightarrow \overline{\mathcal{H}^{1,0}(M)} \xrightarrow{R} H_{\bar{\partial}}^{0,1}(M) \xrightarrow{\partial} H_{BC}^{1,1}(M, \mathbb{R}) \xrightarrow{P} H^2(M)$$

we obtain that $\langle \theta^{0,1} \rangle \oplus \overline{\mathcal{H}^{1,0}(M)} = H_{\bar{\partial}}^{0,1}(M)$.

Step 3: Since the natural map $\tau : H^1(M, \mathbb{R}) \longrightarrow H_{\bar{\partial}}^{0,1}(M)$ is injective, and $H_{\bar{\partial}}^{0,1}(M) = \langle \theta^{0,1} \rangle \oplus \overline{\mathcal{H}^{1,0}(M)}$, we obtain that τ is surjective and $H^1(M, \mathbb{R})$ is generated by the real parts of $\overline{\mathcal{H}^{1,0}(M)}$ and $\langle \theta \rangle$. ■

COROLLARY: Let M be a complex surface. **Then $b_1(M)$ is odd when $\delta(M) = 1$ and $b_1(M)$ is even when $\delta(M) = 0$.**

Proof: When $\delta(M) = 0$, we have $H^1(M, \mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)}$, and when $\delta(M) = 1$, we have $H^1(M, \mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \oplus \langle \theta \rangle$. ■

$H_{\bar{\partial}}^{0,1}(M)$ for a complex surface

COROLLARY: Let M be a complex surface. **Then** $H_{\bar{\partial}}^{0,1}(M) = \overline{\mathcal{H}^{1,0}(M)}$ **when** $\delta(M) = 0$ **and** $H_{\bar{\partial}}^{0,1}(M) = \overline{\mathcal{H}^{1,0}(M)} \oplus \langle \theta^{0,1} \rangle$, where θ is the closed 1-form defined in Proposition 5, and $\theta^{0,1}$ the Dolbeault class of its (0,1)-part.

Proof: See Step 2 in the proof of Proposition 5. ■

Degeneration of the Hodge-de Rham-Frölicher spectral sequence

DEFINITION: Let M be a compact complex manifold. We say that **the Hodge-de Rham-Frölicher spectral sequence degenerates in the term $E_1^{p,q}$** if any class in $H_{\bar{\partial}}^{p,q}(M)$ can be represented by a $\bar{\partial}$ -closed (p,q) -form α which satisfies $\partial\alpha \in \text{im } \bar{\partial}$. It **degenerates on the page $E_1^{*,*}$** if it degenerates for all terms $E_1^{p,q}$.

REMARK: The degeneration in E_1 is equivalent to $E_1^{p,q} = E_{\infty}^{p,q}$, where $E_{*}^{*,*}$ is the Hodge-de Rham-Frölicher spectral sequence. Indeed, $E_1^{p,q} = H_{\bar{\partial}}^{p,q}$, and the differential d_1 takes the Dolbeault class $\alpha \in E_1^{p,q} = H_{\bar{\partial}}^{p,q}(M)$ to an element of $H_{\bar{\partial}}^{p+1,q}(M)$ represented by $(d\alpha)^{p+1,q}$. The same argument also implies that all differentials $d_i, i \geq 1$ vanish. However, if all d_i vanish, the spectral sequence degenerates, giving $E_1^{p,q} = E_{\infty}^{p,q}$; conversely, this equivalence implies that all differentials vanish, and any class in $H_{\bar{\partial}}^{p,q}(M)$ can be represented by a form in $\ker d_1$, that is, by a closed form.

COROLLARY: Let M be a complex surface. **Then the Hodge-de Rham-Frölicher spectral sequence degenerates in $E_1^{0,1}$ and in $E_1^{1,0}$.**

Proof: Indeed, all forms in $\mathcal{H}^{1,0}(M) = E_1^{1,0}$ are closed, and $E_1^{0,1} = \overline{\mathcal{H}^{1,0}(M)} \oplus \langle \theta^{0,1} \rangle$; all forms in $\overline{\mathcal{H}^{1,0}(M)}$ are closed, and $\partial(\theta^{0,1}) = \bar{\partial}(\theta^{1,0})$. ■