Complex surfaces

lecture 13: Currents

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Currents and generalized functions

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M} \left(|f| + |\nabla f| + ... + |\nabla^k f|\right)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function, or **a distribution** is a functional on functions with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies. The space of k-currents is denoted $D^k(M)$. For any current ξ and a function τ with compact support, the pairing $\langle \xi, \tau\rangle$ is often (usually) denoted $\int_M \xi \wedge \tau$. In this situation, τ is called **a test-form**.

REMARK: The pairing between forms and currents is denoted as $\alpha, \tau \mapsto \int_M \alpha \wedge \tau$. Using this notation, **we interpret k-forms on n-manifold as k-currents, that is, as functionals on $n-k$-forms with compact support.**

REMARK: By this definition, **generalized functions are sections of $D^n(M)$**, where $n = \dim_{\mathbb{R}} M$.

Currents and generalized functions

REMARK: There is a natural multiplication $\Lambda^a(M) \wedge D^b(M) \longrightarrow D^{a+b}(M)$, compatible with the product structure on differential forms. However, **there is no product structure on currents.**

REMARK: Any section $V \otimes \xi$ of $\Lambda^n(M) \otimes_{C^\infty M} D^0(M)$, gives a functional on functions with compact support, $f \mapsto \int_M (f\alpha) \wedge \xi$. **This identifies $D^n(M)$ and $\Lambda^n(M) \otimes_{C^\infty M} D^0(M)$.**

REMARK: Given a k-form $\alpha \in \Lambda^{n-k}(M)$ and a section η of $D^0(M) \otimes_{C^\infty M} \Lambda^k(M)$, we can consider η as a section of $\Lambda^n(M) \otimes D^0(M) = D^n(M)$. **This defines a map $D^0(M) \otimes_{C^\infty M} \Lambda^k(M) \longrightarrow D^k(M)$.**

CLAIM: Currents are the same as differential forms with coefficients in generalized functions: the natural map $D^0(M) \otimes_{C^\infty M} \Lambda^k(M) = D^k(M)$ is an isomorphism.

Proof: Locally, we can trivialize $\Lambda^{n-k}(M)$, obtaining $\Lambda^{n-k}(M) \cong (C^\infty M) \otimes_{\mathbb{R}} V$, where V is finite-dimensional. The functionals on $(C^\infty M)^n = (C^\infty M) \otimes_{\mathbb{R}} V$ are identified with the direct sum of $\dim V$ copies of the space of generalized functions, hence the natural map $D^0(M) \otimes_{C^\infty M} \Lambda^k(M) \longrightarrow D^k(M)$ is an isomorphism. ■

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact). Montel spaces are **reflexive** (the map to its double dual with strong topology is an isomorphism).

CLAIM: De Rham differential is continuous on currents, and the Poincaré lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions **REMARK: In the literature, this is sometimes called $(n-p,n-q)$-currents**.

CLAIM: The Poincare and Poincare Dolbeault-Grothendieck lemma hold on (p,q)-currents, and **the d- and $\bar\partial$-cohomology are the same as for forms.**

Positive forms

DEFINITION: A positive (1,1)-form on a complex manifold is a form $\eta \in \Lambda^{1,1}_{\mathbb{R}}(M)$ which satisfies $\eta(x, Ix) \geqslant 0$ for any $x \in TM$.

REMARK: "French positivity". For French, "positive" is the same as "non-negative" for the rest of the world. We will call functions "non-negative" if they are $\geqslant 0$, but if these functions are considered as 0-forms, we have to say they are "positive". **Please don't be confused!**

CLAIM: Let α be a positive function, and u a (1,0)-form. Then $-\sqrt{-1}\,\alpha u \wedge \overline{u}$ is a positive (1,1)-form. Moreover, **any positive form is obtained as a linear combination of such (1,1)-forms.**

Proof: Using the normal form of a positive (1,1)-form on a complex vector space (sometimes known as "polar decomposition"; see Lecture 10, paghe 2), we find that any positive (1,1)-form on an almost complex manifold can be locally represented as $\sum_i -\sqrt{-1}\,\alpha_i u_i \wedge \overline{u}_i$, where $\alpha \geqslant 0$ are non-negative functions, and $u_i \in \Lambda^{1,0}(M)$ an orthonormal frame. ■

Riesz representation theorem

DEFINITION: Let M be a locally compact topological space. A **Borel measure** is a σ-additive measure on Borel sets. A **Radon measure** is a Borel measure which is finite on any compact set.

Riesz representation theorem: Let M be a metrizable, locally compact topological space, $C^0_c(M)$ the space of continuous functions with compact support, and $C^0_c(M)^*$ the space of functionals continuous in uniform topology. **Then the Radon measures on M can can be characterized as functionals $\mu \in C^0_c(M)^*$ which are non-negative on all non-negative functions.**

Proof: Clearly, all measures give such functionals. Conversely, consider a functional $\mu \in C^0_c(M)^*$ which is non-negative on all non-negative functions. Given a closed set $K \subset M$, the characteristic function χ_K can be obtained as a monotonously decreasing limit of continuous functions f_i which are equal to 1 on K **(prove it).** Define $\mu(K) := \lim_i \mu(f_i)$; this limit is well defined because the sequence $\mu(f_i)$ is monotonous. This gives a Radon measure on M **(prove it).** ■

Positive currents

REMARK: From now on, **we assume that $\dim_{\mathbb{C}} M = 2$: we will operate with currents on a complex surface.** The general definitions are slightly more complicated.

REMARK: A positive generalized function multiplied by a positive volume form **gives a measure on a manifold**. By Riesz representation theorem, positive generalized functions are all C^0-continuous as functionals on $C^\infty M$.

DEFINITION: The cone of positive $(1,1)$-currents is generated by $-\sqrt{-1}\alpha u \wedge \overline{u}$, where α is a positive generalized function (that is, a measure), and u a (1,0)-form.

REMARK: For any positive (1,1)-form ρ and a positive current κ, the product $\rho \wedge \kappa$ is a measure; this impies that **a positive current is C^0-continuous**. By Riesz representation theorem, this implies that **a positive current is a (1,1)-form with coefficients in the space of signed measures**.

Positive currents (2)

CLAIM: A product of two positive (1,1)-forms is a positive volume form. Conversely, consider a real (1,1)-form α such that $\alpha \wedge \tau$ is positive for all positive τ. **Then α is also positive.**

Proof: Prove it as an exercise.

This argument immediately implies that **the following definition is equivalent to the one we gave above**.

DEFINITION: A $(1,1)$-current α on a surface is called **positive** if $\int_M \alpha \wedge \tau \geqslant 0$ for any positive $(1,1)$-form τ with compact support.

EXAMPLE: Let C be a compact complex curve on a surface M. **Its current of integration** is $\alpha \mapsto \int_Z \alpha$. It is not hard to see that **this current is closed and positive.**