Complex surfaces

lecture 14: Pseudo-effective cone, Gauduchon cone and Hahn-Banach theorem

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Zorn lemma

DEFINITION: Partial order is a relation $x \prec y$, which is **transitive** (if $x \prec y$ and $y \prec z$ then $x \prec z$) and **non-reflexive** ($z \prec z$ does not hold for any z). A set with partial order is called **partially ordered set**, or **poset**.

DEFINITION: Let $(S, \prec)$ be a poset. An element $x \in S$ is called **maximal** if there is no $y \in S$ such that $x \prec y$. For a subset $S_1 \subset S$ and $x \in S$, we write $S_1 \prec x$ if $\xi \prec x$ for all $\xi \in S_1$.

DEFINITION: A partial order on S is called **linear order**, or **total order** if for all $x \neq y$ either $x \prec y$ or $y \prec x$.

Zorn lemma: Let $(S, \prec)$ be a poset such that for any linearly ordered subset $S_1 \subset S$ there exists $x \in S_1$ such that $S_1 \backslash \{x\} \prec x$. **Then S has a maximal element**.

Hahn-Banach separation theorem

DEFINITION: We say that **a hyperplane** in a topological vector space V is a closed codimension 1 subspace $H \subset V$.

THEOREM: (Hahn-Banach separation theorem)
Let V be a locally convex topological vector space, $A \subset V$ an open convex subset, and $W \subset V$ a closed subspace. Assume that $W \cap A = \emptyset$. **Then there exists a continuous functional $\xi \in V^*$ such that $\xi(W) = 0$ and $\xi(A) > 0$.**

Proof. Step 1: We will prove **existence of a hyperplane $H \subset V$ not intersecting A.** We call this "the absolute version of Hahn-Banach". We will also prove a stronger statement: for any closed subspace $W \subset V$ not intersecting A, the hyperplane $H \subset V$ **can be chosen in such a way that $H \supset W$.** We call this "the relative version of Hahn-Banach theorem". The statement of the theorem immediately follows from the "relative version", because there exists a functional vanishing on H, and hence on W.

Hahn-Banach separation theorem (2)

Proof. Step 1: We will prove **existence of a hyperplane $H \subset V$ not intersecting A.** We call this "the absolute version of Hahn-Banach". We will also prove a stronger statement: for any closed subspace $W \subset V$ not intersecting A, the hyperplane $H \subset V$ **can be chosen in such a way that $H \supset W$.** We call this "the relative version of Hahn-Banach theorem". The statement of the theorem immediately follows from the "relative version", because there exists a functional vanishing on H, and hence on W.
Step 2: We prove now that **the "absolute" case of Hahn-Banach theorem implies the "relative" case.** Let $V' := V/W$ and A', W' the image of A, W in V'. Clearly, $A \not\ni 0$; indeed, if $A' \ni 0$, this would imply that $A \cap W \neq \emptyset$. The "absolute" case of Hahn-Banach applied to V', A' gives a closed hyperplane H' not intersecting A'; its preimage in V is a hyperplane H, $H \cap A = \emptyset$.
Step 3: Prove the Hahn-Banach theorem when $V = \mathbb{R}^2$ and $W = 0$ as an exercise (you need to find a line in $\mathbb{R}^2$ not intersecting a given convex cone $A \subset \mathbb{R}^2$, $A \not\ni 0$.)
Step 4: Let $V_0 \subset V$ be a hyperplane, and $A \subset V$ an open convex cone, $A \not\ni 0$. Suppose that in V_0 Hahn-Banach theorem is already proven, and V_0 contains a hyperplane H_0 not intersecting A. Then V/H_0 is 2-dimensional, and the image of A in V/H_0 is an open cone not containing 0, hence **(the absolute) Hahn-Banach theorem for (V, A) follows from the absolute Hahn-Banach theorem for $(V_0, A \cap V_0)$.**

Hahn-Banach separation theorem (3)

Step 5: We can now prove (the absolute) Hahn-Banach theorem using the Zorn lemma. Fix (V, A). Consider the following poset $\mathfrak{S}$: its elements are closed subspaces $V_1 \subset V$ together with a hyperplane $H_1 \subset V_1$ such that $A \cap H_1 = 0$. The order is defined in a natural way, $(V_1, H_1) \succ (V_2, H_2)$ if $V_1 \supset V_2$ and $H_2 = H_1 \cap V_2$. Consider a linear ordered chain $\{(V_\alpha, H_\alpha)\}$ of elements in $\mathfrak{S}$. Clearly, $H_0 := \overline{\bigcup H_\alpha}$ is a closed hyperplane in $V_0 := \overline{\bigcup V_\alpha}$, not intersecting A. Since A is open, the closure of H_0 is a hyperplane of V_0 not intersecting A. Therefore, any linear ordered chain in $\mathfrak{S}$ has a maximal element. Applying the Zorn lemma, we obtain a maximal element $(V_{\mathsf{max}}, H_{\mathsf{max}}) \in \mathfrak{S}$. **It remains to show that $V_{\mathsf{max}} = V$.**

Step 6: Consider the quotient $V' := V/V_{\mathsf{max}}$, and let A' be the image of A in V'. Since $A \not\supset V_{\mathsf{max}}$, the cone A' does not contain 0, hence there exists a line $l \subset V'$ such that $l \cap A'$ is a ray in l. Take $V_1 := V_{\mathsf{max}} + \langle l \rangle$ and apply the 2-dimensional (absolute) Hahn-Banach theorem to $V_2 := V_1/H_{\mathsf{max}} = \mathbb{R}^2$, and A_2 the image of $A \cap V_1$ in V_2. The set A_2 does not contain zero, because $A \cap H_{\mathsf{max}} = \emptyset$. The finite-dimensional Hahn-Banach theorem gives us a hyperplane $H_2 \subset V_2$ not intersecting A_2. Let $H_1 \subset V_1$ be its preimage. **Then $H_1 \cap A = 0$ and $(V_1, H_1) \succ (V_{\mathsf{max}}, H_{\mathsf{max}})$, giving a contradiction.** ■

Currents and generalized functions (reminder)

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M}\Big(|f| + |\nabla f| + ... + |\nabla^k f|\Big)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function, or **a distribution** is a functional on functions with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies. The space of k-currents is denoted $D^k(M)$. For any current ξ and a function τ with compact support, the pairing $\langle \xi, \tau\rangle$ is often (usually) denoted $\int_M \xi\wedge\tau$. In this situation, τ is called **a test-form**.

Currents are differential forms with coefficients in generalized functions (reminder)

REMARK: The pairing between forms and currents is denoted as $\alpha, \tau \mapsto \int_M \alpha \wedge \tau$. Using this notation, **we interpret k-forms on n-manifold as k-currents, that is, as functionals on $n-k$-forms with compact support.**

CLAIM: Currents are the same as differential forms with coefficients in generalized functions: the natural map $D^0(M) \otimes_{C^\infty M} \Lambda^k(M) = D^k(M)$ is an isomorphism.

Proof: Locally, we can trivialize $\Lambda^{n-k}(M)$, obtaining $\Lambda^{n-k}(M) \cong (C^\infty M) \otimes_{\mathbb{R}} V$, where V is finite-dimensional. The functionals on $(C^\infty M)^n = (C^\infty M) \otimes_{\mathbb{R}} V$ are identified with the direct sum of $\dim V$ copies of the space of generalized functions, hence the natural map $D^0(M) \otimes_{C^\infty M} \Lambda^k(M) \longrightarrow D^k(M)$ is an isomorphism. ■

Currents on complex manifolds (reminder)

CLAIM: De Rham differential is continuous on currents, and the Poincaré lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions.

REMARK: In the literature, this is sometimes called $(n-p,n-q)$-currents.

CLAIM: The Poincare and Poincare Dolbeault-Grothendieck lemma hold on (p,q)-currents, and **the d- and $\bar\partial$-cohomology are the same as for forms.**

DEFINITION: A positive (1,1)-form on a complex manifold is a form $\eta\in \Lambda^{1,1}_{\mathbb R}(M)$ which satisfies $\eta(x,Ix)\geqslant 0$ for any $x\in TM$.

CLAIM: Let α be a positive function, and u a (1,0)-form. Then $-\sqrt{-1}\,\alpha u\wedge\overline{u}$ is a positive (1,1)-form. Moreover, **any positive form is obtained as a linear combination of such (1,1)-forms.**

Positive currents

REMARK: From now on, **we assume that** $\dim_{\mathbb{C}} M = 2$**: we will operate with currents on a complex surface.** The general definitions are slightly more complicated.

REMARK: A positive generalized function multiplied by a positive volume form **gives a measure on a manifold**. By Riesz representation theorem, positive generalized functions are all C^0-continuous as functionals on $C^\infty M$.

CLAIM: The following definitions are equivalent.

DEFINITION: The cone of positive $(1,1)$-currents is generated by $-\sqrt{-1}\alpha u \wedge \overline{u}$, where α is a positive generalized function (that is, a measure), and u a (1,0)-form.

DEFINITION: A $(1,1)$-current α on a surface is called **positive** if $\int_M \alpha \wedge \tau \geqslant 0$ for any positive $(1,1)$-form τ with compact support.

EXAMPLE: Let C be a compact complex curve on a surface M. **Its current of integration** is $\alpha \mapsto \int_Z \alpha$. It is not hard to see that **this current is closed and positive.**

Aeppli cohomology

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{AE}(M)$ the space of dd^c-closed (p,q)-forms modulo $\partial(\Lambda^{p-1,q}M) + \overline{\partial}(\Lambda^{p,q-1}M)$. Then $H^{p,q}_{AE}(M)$ is called **the Aeppli cohomology** of M.

THEOREM: (A. Aeppli)
Let M be a compact complex n-manifold. Then **the Aeppli cohomology is finite-dimensional.** Moreover, the natural pairing

$$H^{p,q}_{BC}(M) \times H^{n-p,n-q}_{AE}(M) \longrightarrow H^{2n}(M) = \mathbb{C},$$

taking x, y to $\int_M x \wedge y$ **is non-degenerate and identifies $H^{p,q}_{BC}(M)$ with the dual $H^{n-p,n-q}_{AE}(M)^*$.**

Proof: Use the same argument as used to prove Serre's duality and Poincaré duality. ■

REMARK: Math Genealogy knows 3 person called Aeppli: Alfred Aeppli (ETH Zürich, 1924, student of George Pólya and Hermann Weyl), Alfred Aeppli (ETH Zürich, 1956, student of Beno Eckmann and Heinz Hopf), and Hans Aeppli (1980, student of Hans Storrer). *The second Alfred Aeppli was the one responsible for Aeppli cohomology.*

DEFINITION: Let M be a compact complex n-manifold. Its **Gauduchon cone** is the set of all Aeppli classes of ω^{n-1}, where ω is a Gauduchon metric.

Pseudo-effective cone

DEFINITION: A Bott-Chern cohomology class $\eta \in H^{1,1}_{BC}(M,\mathbb{R})$. is called **pseudo-effective** if it can be represented by a positive, closed current. **The pseudo-effective cone** is the set of all pseudo-effective classes.

REMARK: Let ω be a Gauduchon metric on a complex n-manifold, and η a closed $(1,1)$-form. We define **the degree** $\deg_\omega \eta := \int_M \eta \wedge \omega^{n-1}$. Since $\int_M dd^c f \wedge \omega^{n-1} = 0$, this number depends only on the cohomology class $[\eta] \in H^{1,1}_{BC}(M,\mathbb{R})$.

The next theorem claims that **the Gauduchon cone is dual to the pseudo-effective cone.**

Pseudo-effective cone

THEOREM: (Ahcène Lamari)
Let M be a compact complex manifold. A class $\eta \in H^{1,1}_{BC}(M, \mathbb{R})$ **is pseudo-effective if and only if $\deg_\omega \eta > 0$ for any Gauduchon metric ω.**

Proof. Step 1: Let A be the set of strictly positive $(n-1, n-1)$-forms, $u \in H^{n-1,n-1}_{AE}(M, \mathbb{R})$, and $V = u + V'$, where V' is the space of all $(n-1, n-1)$-parts of $2n-2$-forms. Clearly, $V \cap A = \emptyset$ if and only if u is not in the Gauduchon cone. By Hahn-Banach, **$V \cap A = \emptyset$ if and only if there exists a functional on $\Lambda^{n-1,n-1}(M)$ (that is, a (1,1)-current) ξ such that $\langle \xi, A \rangle > 0$ and $\langle \xi, V \rangle = 0$.**

Step 2: The condition $\langle \xi, A \rangle > 0$ **means that ξ is a non-zero positive current.** The condition $\langle \xi, V' \rangle = 0$ **is equivalent to $\int \xi \wedge dw = 0$ for all $(2n-3)$-forms w.** Indeed, V' is the space of $(n-1, n-1)$-parts of exact forms, and ξ is a (1,1)-current.

Step 3: However, $\int \xi \wedge dw = 0$ for all w is equivalent to ξ being closed. Then, **a class $u \in H^{n-1,n-1}_{AE}(M, \mathbb{R})$ belongs to the Gauduchon cone if and only if $\int_M \xi \wedge u > 0$ for all positive, closed (1,1)-currents ξ.** ■