Complex surfaces

lecture 15: Plurisubharmonic and subharmonic functions

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Plurisubharmonic functions

DEFINITION: A function f on a complex manifold is called **plurisubharmonic** (or **psh**) if $dd^c f$ is a positive (1,1)-form, and **strictly plurisubharmonic** if $dd^c f$ is a positive definite (and ipso facto Kähler) form.

REMARK: For any plurisubharmonic function f, and any Hermitian form ω, we have $\Delta(f) \geqslant 0$, where Δ is an elliptic operator. Applying the strong maximum principle, we obtain
COROLLARY: A plurisubharmonic function on a manifold **cannot have a local maximum, unless it is constant.** ■

EXAMPLE: A sum of plurisubharmonic functions is plurisubharmonic.
EXAMPLE: Let f be a holomorphic function. Then $dd^c|f|^2 = 2\sqrt{-1}\,\partial\bar\partial f\bar{f} = 2\sqrt{-1}\,(\partial f \wedge \overline{\partial f})$, hence $|f|^2$ is plurisubharmonic.

COROLLARY: Let $f_1, ..., f_n$ be a collection of holomorphic functions on a complex manifold. **Then $\sum_i |f_i|^2$ is plurisubharmonic, hence it cannot have a maximum.**
EXAMPLE: Let $\mu \in C^\infty\mathbb{R}$. Then

$$dd^c(\mu(f))|_m = \mu'(f(z))^2 dd^c f + \mu''(f(z)) df \wedge d^c f.$$

Therefore, **for any psh function f, the composition $\mu(f)$ is psh when $\mu'' \geqslant 0$ and $\mu' > 0$.**

Kähler potential

I will use the following difficult theorem without a proof.

LEMMA: ("Poincaré-Dolbeault-Grothendieck lemma")
Let η be a $\overline{\partial}$-closed (p,q)-form, $q > 0$, on an open ball $B \subset \mathbb{C}^n$. **Then $\eta \in \operatorname{im} \overline{\partial}$.**

DEFINITION: A closed Hermitian (1,1)-form ω is called **a Kähler form**. A function f is called **its Kähler potential** when $\omega = dd^c f$.

CLAIM: Any Kähler form on an open ball $B \subset \mathbb{C}^n$ **admits a Kähler potential.** Moreover, for any closed (1,1)-form η on B **there exists a function $f \in C^\infty B$ such that $\eta = dd^c f$.**

Proof. Step 1: Poincaré-Dolbeault-Grothendieck lemma implies that $\omega = \overline{\partial}\eta$, for some $\eta \in \Lambda^{1,0} B$. Then $\partial\overline{\partial}\eta = \partial\omega = 0$, which implies $\overline{\partial}\partial\eta = 0$.

Step 2: We obtain that $\partial\eta$ is a holomorphic $(2,0)$-form, which is closed, because $\partial^2\eta = \overline{\partial}\partial\eta = 0$. Applying the Poincaré lemma as above, **we obtain that $\partial\eta = d\alpha$, where α is a holomorphic (1,0)-form.**

Step 3: Now, $\partial(\eta - \alpha) = 0$, which implies that $\eta - \alpha \in \operatorname{im} \partial$ by Poincaré-Dolbeault-Grothendieck lemma again. Take f such that $\partial f = \eta - \alpha$. Since α is holomorphic, we have $\overline{\partial}(\eta - \alpha) = \overline{\partial}\eta = \omega$. **This brings $\omega = \overline{\partial}\partial f$.** ■

Subharmonic functions

DEFINITION: (René Baire, 1899)
A function $f: M \longrightarrow \mathbb{R}$ on a topological space is called **upper semicontinuous** if $\limsup_{z \to z_0} f(z) \leqslant f(z_0)$.

DEFINITION: An upper semicontinuous measurable function f on an open set ("domain") $\Omega \subset \mathbb{R}^n$ is called **subharmonic** if for any ball $B \subset \Omega$ with center in z_0, we have $f(z_0) \leqslant \operatorname{Av}_{z \in B} f(z)$.

REMARK: From the exercises in the handout 3, we obtain that **this is equivalent to $\Delta(f) \geqslant 0$,** where Δ is the Laplacian on generalized functions.

EXERCISE: Prove directly that **a subharmonic function which attains a local maximum on a connected domain $\Omega \subset \mathbb{R}^n$ is constant.**

CLAIM: For any decreasing sequence $\{u_k\}$ of subharmonic functions, **the limit $u := \lim u_k$ is also subharmonic.**

Proof: The decreasing limit is upper semicontinuous, and the inequality $f(z_0) \leqslant \operatorname{Av}_{z \in B} f(z)$ remains true by Lebesgue's monotone convergence theorem. ■

Composing subharmonic functions and convex functions

THEOREM: Let $u_1, ...u_p$ be subharmonic functions on $\Omega \subset \mathbb{R}^n$, and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ a convex function which is monotonously non-decreasing in each variable. **Then $\chi(u_1, ..., u_p)$ is also subharmonic.**

Proof: Every convex function is continuous, hence $\chi(u_1, ..., u_p)$ is upped semi-continuous. Clearly, $\chi(t) = \sup_{A \in \mathbb{A}} A_i(t)$, where $\mathbb{A}$ is the set of affine functions $A(t_1, ..., t_p) = \sum a_i t_i + b$ such that $A(x) \leqslant \chi(x)$. Since χ is non-decreasing in each variable, we have $a_i \geqslant 0$ for $i = 1, ..., p$, which gives

$$\sum a_i u_i(z_0) + b \leqslant \operatorname{Av}_{z \in B} \left(\sum a_i u_i(z) + b\right) \leqslant \operatorname{Av}_{z \in B} \chi(u_1(z), ..., u_p(z)).$$

■

COROLLARY: This implies that $\sum u_i$, $\max(u_1, ..., u_p)$, $\log(\sum e^{u_i})$ **are subharmonic, if u_i are subharmonic.**

Proof: For the last assertion, we need to check that the function $t_1, ..., t_p \mapsto \log(\sum e^{t_i})$ is convex. Let $E_x := (e^{1/2\, x_1}, ..., e^{1/2\, x_p})$, $E_y := (e^{1/2\, y_1}, ..., e^{1/2\, y_p})$. The Cauchy-Schwarz inequality $|E_x||E_y| \geqslant (E_x, E_y)$ gives

$$\left(\sum_i e^{x_i}\right)^{\frac{1}{2}} \left(\sum_i e^{y_i}\right)^{\frac{1}{2}} \geqslant \left(\sum_i e^{\frac{x_i + y_i}{2}}\right).$$

Taking logarithms, obtain

$$\frac{1}{2} \log \sum_i e^{x_i} + \frac{1}{2} \log \sum_i e^{y_i} \geqslant \log \sum_i e^{\frac{x_i + y_i}{2}},$$

which is the same as $\frac{\chi(x) + \chi(y)}{2} \geqslant \chi\left(\frac{x+y}{2}\right)$; this is equivalent to convexity. ■

Pullback and pushforward

DEFINITION: Let $f: X \longrightarrow Y$ be a proper holomorphic map of complex manifolds, $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y + k$, and α a (p,q)-current on X. Define **the pushforward** $f_* \alpha$ using $\langle f_*\alpha, \tau\rangle := \langle \alpha, f^*\tau\rangle$, where τ is any test-form. Then **$f_*\alpha$ has bidimension $(p-k, q-k)$.** One should think of f_* as of fiberwise integration.

REMARK: Clearly, $df_*\alpha = f_* d\alpha$, $\partial f_*\alpha = f_*\partial\alpha$, and so on.

REMARK: One defines pushforward for real manifolds in the same way.

REMARK: Pullback of currents **is (generally speaking) not well-defined.** Pushforward of differential forms are often **singular**, that is, are given by a current. However...

CLAIM: Let $f: X \longrightarrow Y$ be a proper submersion of smooth manifolds. **Then a pushforward of a differential form is a differential form.**

Proof: Indeed, the pushforward **can be understood as a fiberwise integral.**

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Example 1: Let $\pi_1, \pi_2, M \times M \longrightarrow M$ be projection maps, and $\delta_\Delta \in D^{\dim M}(M)$ the integration current for the diagonal $\Delta \subset M \times M$. **Then $(\pi_2)_*(\pi_1^*(\eta) \wedge \delta_\Delta) = \eta$ for any differential form η on M.** Indeed, for any $\tau \in \Lambda_c^k M$, $\eta \in \Lambda^{\dim M - k}(M)$ we have

$$\int_{M\times M} \pi_1^*\eta \wedge \delta_\Delta \wedge \pi_2^*\tau = \int_\Delta \pi_1^*\eta \wedge \pi_2^*\tau = \int_M \eta \wedge \tau.$$