Complex surfaces

lecture 16: Integral transform

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Pullback and pushforward (reminder)

DEFINITION: Let $f: X \longrightarrow Y$ be a proper holomorphic map of complex manifolds, $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y + k$, and α a (p,q)-current on X. Define **the pushforward** $f_* \alpha$ using $\langle f_*\alpha, \tau\rangle := \langle \alpha, f^*\tau\rangle$, where τ is any test-form. Then **$f_*\alpha$ has bidimension $(p-k, q-k)$.** One should think of f_* as of fiberwise integration.

REMARK: Clearly, $df_*\alpha = f_* d\alpha$, $\partial f_*\alpha = f_*\partial\alpha$, and so on.

REMARK: One defines pushforward for real manifolds in the same way.

REMARK: Pullback of currents **is (generally speaking) not well-defined.** Pushforward of differential forms are often **singular**, that is, are given by a current. However...

CLAIM: Let $f: X \longrightarrow Y$ be a proper submersion of smooth manifolds. **Then a pushforward of a differential form is a differential form.**

Proof: Indeed, the pushforward **can be understood as a fiberwise integral.**

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Example 1: Let $\pi_1, \pi_2, M\times M \longrightarrow M$ be projection maps, and $\delta_\Delta \in D^{\dim M}(M)$ the integration current for the diagonal $\Delta \subset M \times M$. **Then $(\pi_2)_*(\pi_1^*(\eta)\wedge \delta_\Delta) = \eta$ for any differential form η on M.** Indeed, for any $\tau \in \Lambda^k_c M$, $\eta \in \Lambda^{\dim M - k}(M)$ we have

$$\int_{M\times M} \pi_1^*\eta \wedge \delta_\Delta \wedge \pi_2^*\tau = \int_\Delta \pi_1^*\eta\wedge\pi_2^*\tau = \int_M \eta\wedge\tau.$$

Kernels of integration

DEFINITION: Let $\pi_1, \pi_2, \pi_\Sigma : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ take (x, y) to: $\pi_1(x, y) = x$, $\pi_2(x, y) = y$, $\pi_\Sigma(x, y) = x - y$. Consider a differential form η on $\mathbb{R}^n$, and let V be a measure with compact support on $\mathbb{R}^n$, called **kernel of integration**. The **convolution with the kernel V** is

$$\eta \star V := (\pi_2)_*(\pi_1^*(\eta) \wedge \pi_\Sigma^* V)$$

PROPOSITION: $\eta \star V = \int_{x \in \mathbb{R}^n} L_x^*(\eta) V$, where L_x is a translation on $x \in \mathbb{R}^n$.

Proof. Step 1: Let $x_1, ..., x_n$ be coordinates on the first component, $y_1, ..., y_n$ the coordinates on the second component, and $\rho = dx_{i_1} \wedge ... \wedge dx_{i_k}$ a coordinate monomial on dx_i. Without restricting the generality, we may assume $\eta = f\rho$, where f is a smooth function on $\mathbb{R}^n$. Let $V = g\,\mathrm{Vol}$, where g is a function on $\mathbb{R}^n$ with compact support. Then

$$\pi_1^*(\eta) \wedge \pi_\Sigma^*(V) = f(x_1, ..., x_n) g(x_1 - y_1, ..., x_n - y_n) \rho \wedge \prod_{i=1}^n dx_i - dy_i.$$

Kernels of integration (2)

Proof. Step 1: Let $x_1, ..., x_n$ be coordinates on the first component, $y_1, ..., y_n$ the coordinates on the second component, and $\rho = dx_{i_1} \wedge ... \wedge dx_{i_k}$ a coordinate monomial on dx_i. Without restricting the generality, we may assume $\eta = f\rho$, where f is a smooth function on $\mathbb{R}^n$. Let $V = g\,\mathrm{Vol}$, where g is a function on $\mathbb{R}^n$ with compact support. Then

$$\pi_1^*(\eta) \wedge \pi_\Sigma^*(V) = f(x_1, ..., x_n) g(x_1 - y_1, ..., x_n - y_n) \rho \wedge \prod_{i=1}^n (dx_i - dy_i).$$

Step 2: To take a pushforward $(\pi_2)_* \left(h\rho \wedge \prod_{i=1}^n dx_i - dy_i\right)$, we decompose this form into a sum of monomials and drop all monomials which don't have a form $hdy_{j_1} \wedge ... \wedge dy_{j_k} \wedge \prod_{i=1}^n dx_i$ and integrate $h \prod_{i=1}^n dx_i$. However, the only monomial component of $dx_{i_1} \wedge ... \wedge dx_{i_k} \wedge \prod_{i=1}^n (dx_i - dy_i)$ which has form $dy_{j_1} \wedge ... \wedge dy_{j_k} \wedge \prod_{i=1}^n dx_i$ is $dy_{i_1} \wedge ... \wedge dy_{i_k} \wedge \prod_{i=1}^n dx_i$. Then $(\pi_2)_* \left(h\rho \wedge \prod_{i=1}^n (dx_i - dy_i)\right)$ evaluated at $(y_1, ..., y_n) \in \mathbb{R}^n$ is $\int_{\mathbb{R}^n} f(x_1, ..., x_n) g(x_1 - y_1, ..., x_n - y_n) dy_{i_1} \wedge ... \wedge dy_{i_k} \wedge \prod_{i=1}^n dx_i$. This integral is equal to

$$\int_{\mathbb{R}^n} f(x_1 + y_1, ..., x_n + y_n) g(x_1, ..., x_n) dy_{i_1} \wedge ... \wedge dy_{i_k} \wedge \prod_{i=1}^n dx_i =$$

$$= \int_{\mathbb{R}^n} f(x_1 + y_1, ..., x_n + y_n) g(x_1, ..., x_n) dy_{i_1} \wedge ... \wedge dy_{i_k} \wedge \prod_{i=1}^n dx_i = \int_{R^n} L^*_{(x_1, ..., x_n)}(\eta) \wedge V. \blacksquare$$

Smoothing kernels

DEFINITION: Let μ_i be a sequence of non-negative smooth functions on $\mathbb{R}^n$ with support in an open ball B_{r_i} with center in 0, such that r_i converges to 0, and $\int_{\mathbb{R}}^n \mu_i \operatorname{Vol} = 1$. The family $\mu_i \operatorname{Vol}$ is called **the family of smoothing kernels.**

REMARK: Let $\pi_1, \pi_2, \pi_\Sigma : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ take (x,y) to: $\pi_1(x,y) = x$, $\pi_2(x,y) = y$, $\pi_\Sigma(x,y) = x - y$. Consider a differential form η on $\mathbb{R}^n$. Its **convolution with smoothing kernels** is

$$\eta \star \mu_i := (\pi_2)_*(\pi_1^*(\eta) \wedge \pi_\Sigma^*(\mu_i \operatorname{Vol}))$$

CLAIM: For any η with integrable coefficients, $\eta \star \mu_i$ **is smooth, and** $\lim_i(\eta \star \mu_i) = \eta$. Also, **the map** $\eta \mapsto \eta \star \mu_i$ **commutes with de Rham differential,** and (if we are operating over $\mathbb{C}^n$) with d^c and Dolbeault differentials,

Proof: The limit $\lim_i(\eta \star \mu_i) = \eta$ is clear, because $\lim \pi_\Sigma^*(\mu_i \operatorname{Vol}) = \delta_\Delta$, hence $\lim_i(\eta \star \mu_i) = (\pi_2)_*(\pi_1^*(\eta) \wedge \delta_\Delta) = \eta$ (Example 1). For smoothness of $\eta \star \mu_i$, we notice that $\frac{d}{dx_i}\eta \star \mu_i = \eta \star \frac{d}{dx_i}\mu_i$. This operation commutes with de Rham differential, because $\pi_\Sigma^*(\mu_i \operatorname{Vol})$ is closed, hence $(\pi_2)_*(\pi_1^*(d\eta) \wedge \pi_\Sigma^*(\mu_i \operatorname{Vol})) = (\pi_2)_*(d(\pi_1^*(\eta)) \wedge \pi_\Sigma^*(\mu_i \operatorname{Vol})) = d((\pi_2)_*(\pi_1^*(\eta) \wedge \pi_\Sigma^*(\mu_i \operatorname{Vol})))$.

REMARK: To see that $\eta \mapsto \eta \star \mu_i$ commutes with the de Rham differential, we can also **note that** $\eta \star \mu_i = \int_{x \in \mathbb{R}^n} \mu_i(x) L_x \eta$**, where** L_x **is a translation on** $x \in \mathbb{R}^n$**.**

Subharmonic functions (reminder)

DEFINITION: (René Baire, 1899)
A function $f : M \longrightarrow \mathbb{R}$ on a topological space is called **upper semicontinuous** if $\limsup_{z\to z_0} f(z) \leqslant f(z_0)$.

DEFINITION: An upper semicontinuous measurable function f on an open set ("domain") $\Omega \subset \mathbb{R}^n$ is called **subharmonic** if for any ball $B \subset \Omega$ with center in z_0, we have $f(z_0) \leqslant \operatorname{Av}_{z\in B} f(z)$.

REMARK: From the exercises in the handout 3, we obtain that **this is equivalent to $\Delta(f) \geqslant 0$,** where Δ is the Laplacian on generalized functions.

EXERCISE: Prove directly that **a subharmonic function which attains a local maximum on a connected domain $\Omega \subset \mathbb{R}^n$ is constant.**

CLAIM: For any decreasing sequence $\{u_k\}$ of subharmonic functions, **the limit $u := \lim u_k$ is also subharmonic.**

Proof: The decreasing limit is upper semicontinuous, and the inequality $f(z_0) \leqslant \operatorname{Av}_{z\in B} f(z)$ remains true by Lebesgue's monotone convergence theorem. ■

Average of a subharmonic function over a ball

THEOREM: Let v be a subharmonic function on a ball $B_1 \subset \mathbb{R}^n$ centered in 0. **Then the average $\operatorname{Av}_{B_r} v$ monotonously decreases to $v(0)$** as r decreases to 0.

Proof: From Assignment 3:

$$\int_{B_r} u\Delta v \operatorname{Vol}_{B_r} - \int_{B_r} v\Delta u \operatorname{Vol}_{B_r} = \int_{S_r} (u \operatorname{Lie}_\nu v - v \operatorname{Lie}_\nu u) \operatorname{Vol}_{S_r}$$

where B_r is a ball of radius r, S_r its boundary, and ν is a unit radial vector (normal to the sphere S_r). Applied to $u = 1$, this gives

$$\int_{B_r} \Delta v \operatorname{Vol}_{B_r} = \int_{S_r} \operatorname{Lie}_\nu v \operatorname{Vol}_{S_r} \geqslant 0.$$

However, $\frac{d}{dr} \operatorname{Av}_{S_r}(v) = \operatorname{Av}_{S_r} \operatorname{Lie}_\nu(v)$, hence $\operatorname{Av}_{S_r}(v)$ is monotonous as a function of r. On the other hand,

$$\operatorname{Av}_{B_r}(v) = \frac{\int_0^r t^{n-1} \operatorname{Av}_{S_t}(v)dt}{\int_0^r t^{n-1}dt} = \frac{\int_0^r \operatorname{Av}_{S_t}(v)d(t^n)}{\int_0^r d(t^n)};$$

Since $\operatorname{Av}_{S_t}(v)$ monotonously decreases to $v(0)$ as t decreases to 0, this implies that $\operatorname{Av}_{B_r} v$ also monotonously decreases to $v(0)$ as t decreases to 0. ■

Smoothing kernels and subharmonic functions

REMARK: Let μ_1 be a non-negative smooth function on $\mathbb{R}^n$ which satisgies $\int_{\mathbb{R}}^n \mu_1 \operatorname{Vol} = 1$, has support on a unit ball, and satisfies $\mu_1(z) = \mu(|z|)$, where r is monotonous. Define $\mu_\varepsilon(z) := \varepsilon^{-n}\mu_1(\varepsilon^{-1}z)$. Clearly, we have $\lim_{\varepsilon\to 0} \mu_\varepsilon \operatorname{Vol} = \delta_0$, **hence μ_ε is a family of smoothing kernels.**

THEOREM: Let u be a subharmonic function. Then $u \star \mu_\varepsilon$ **decreases monotonously as a function of ε, and converges to u.** In particular, **all subharmonic functions are obtained as monotonous limits of smooth subharmonic functions.**

Proof: The value of $u\star\mu_\varepsilon$ in x is obtained by averaging u in concentric spheres around x and summing it up with coefficients determined by μ,

$$u \star \mu_\varepsilon(x) = \int_{t=0}^{\varepsilon} dt \cdot \mu(\varepsilon^{-1}t)t^{n-1} \operatorname{Av}_{S_{\varepsilon t}}(u),$$

where S_r is the sphere of radius r centered in x. Since all monotonous functions μ can be obtained by summing up the step functions, it would suffice to show monotonicity when μ is a characteristic function of an interval $[0, r]$. In this case, $u \star \mu_\varepsilon(x)$ is the average of u in a ball of radius εr, which is monotonous as a function of ε by the Theorem 1 above. ■

REMARK: This can be used to show that any generalized function f which satisfies $\Delta(f) \geqslant 0$ **is a limit of a monotonous sequence of smooth subharmonic functions;** in particular, **such function is upper semicontinuous and locally integrable, unless $f = -\infty$ identically.**

Plurisubharmonic functions: singular case

DEFINITION: (Lelong, Oka, 1942)
A function $f: M \longrightarrow [-\infty, \infty[$ on a complex manifold M is called **plurisubharmonic** (psh) if u is upper semicontinuous, and its restriction to any complex line is subharmonic.

From the results on subharmonic functions obtained above, we obtain the following assertions.

CLAIM: For any decreasing sequence $\{u_k\}$ of plurisubharmonic functions, **the limit $u := \lim u_k$ is also subharmonic.**

CLAIM: Let $u_1, ...u_p$ be plurisubharmonic functions on $\Omega \subset \mathbb{R}^n$, and $\chi : \mathbb{R}^p \longrightarrow \mathbb{R}$ a convex function which is monotonously non-decreasing in each variable. **Then $\chi(u_1, ..., u_p)$ is also plurisubharmonic.** This implies that $\sum u_i$, $\max(u_1, ..., u_p)$, $\log\left(\sum e^{u_i}\right)$ **are plurisubharmonic, if u_i are plurisubharmonic.**

CLAIM: Let f be a plurisubharmonic function on a ball $B \subset \mathbb{C}^n$, and $f \star \mu_\varepsilon$ its convolution with the smoothing kernel defined above. **Then $f \star \mu_\varepsilon$ is a monotonous sequence of smooth plurisubharmonic functions, converging to f.** This implies that f is locally integrable, unless $f = -\infty$ identically.

REMARK: This argument also can be applies to generalized functions f such that $dd^c f$ is positive; **a posteriori, f is upper semicontinuous, because f is a monotonous limit of smooth functions.**