Complex surfaces

lecture 17: Poincaré-Lelong formula and regularization of currents

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Cauchy formula

PROPOSITION: (Cauchy formula)
Let f be a smooth function on a unit disk $\Delta \subset \mathbb{C}$. **Then**

$$f(w) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(z)}{z-w} dz - \int_{\Delta} \frac{1}{\pi(z-w)} \frac{\partial f}{\partial \bar z} \mathrm{Vol}, \qquad (*)$$

where $\mathrm{Vol} = dx \wedge dy$ is the standard volume form.

Proof. Step 1: For any subset K of $\mathbb{C}$ with smooth boundary, not containing w,

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial K} \frac{f(z)}{z-w} dz = \int_{K} \frac{1}{\pi(z-w)} \frac{\partial f}{\partial \bar z} \mathrm{Vol}$$

by Stokes' formula, because $d\left(\frac{f(z)}{z-w} dz\right) = -\frac{\partial f}{\partial \bar z}\frac{1}{(z-w)} dz \wedge d\bar z$. Therefore, (*) remains true if we replace a disk by any other open set, containing w, such as a disk with center in w. **Therefore, it suffices to prove (*) when $w = 0$.**

Cauchy formula (2)

PROPOSITION: (Cauchy formula)
Let f be a smooth function on a unit disk $\Delta \subset \mathbb{C}$. **Then**

$$f(w) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(z)}{z-w} dz - \int_{\Delta} \frac{1}{\pi(z-w)} \frac{\partial f}{\partial \overline{z}} \operatorname{Vol}, \qquad (*)$$

Proof. Step 1: ...it suffices to prove (*) when $w=0$.

Step 2: Let Δ_ε be a disk of radius ε centered in 0. From Step 1, it follows that

$$\frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_\varepsilon} \frac{f(z)}{z} dz - \int_{\Delta_\varepsilon} \frac{1}{\pi(z)} \frac{\partial f}{\partial \overline{z}} \operatorname{Vol}$$

is independent on ε. Therefore, to prove (*) it would suffice to show that

$$f(0) = \lim_{\varepsilon \to 0} \left[\frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta_\varepsilon} \frac{f(z)}{z} dz - \int_{\Delta_\varepsilon} \frac{1}{\pi(z)} \frac{\partial f}{\partial \overline{z}} \operatorname{Vol}.\right]$$

The second integral converges to zero, because the function $\frac{1}{z}$ is locally integrable on $\mathbb{C}$ (its integral on any circle centered in 0 is 2π). The first integral gives

$$\int_0^{2\pi} \frac{f(\varepsilon e^{\sqrt{-1}\,\theta})}{\varepsilon e^{\sqrt{-1}\,\theta}} d(\varepsilon e^{\sqrt{-1}\,\theta}) = \int_0^{2\pi} f(\varepsilon e^{\sqrt{-1}\,\theta}) d\theta. \blacksquare$$

Poincaré-Lelong formula on $\mathbb{C}$

PROPOSITION: (Poincaré-Lelong formula on $\mathbb{C}$)
Consider the funcion $l(z) := \log|z|$ on a disk $\Delta \subset \mathbb{C}$. **Then l is plurisubharmonic, and $\frac{1}{4\pi} dd^c l$ is equal to the δ-function δ_0.**

Proof. Step 1: Since $\partial\overline{\partial}\log(z\overline{z}) = \partial\overline{\partial}\log z + \partial\overline{\partial}\log\overline{z}$, the current $dd^c l = 0$ vanishes everywhere outside of 0. Then $\langle dd^c l, f\rangle = 0$ unless the support of f contains 0. Therefore, **it suffices to evaluate the current $dd^c l$ on a function with support in an arbitrary small open neighbourhood of 0.**

Step 2: Take f with support in Δ and apply the Cauchy formula obtained in the previous slide. It gives

$$f(0) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(z)}{z} dz - \int_\Delta \frac{1}{\pi(z)} \frac{\partial f}{\partial\overline{z}} \operatorname{Vol},$$

the first term vanishes because $f = 0$ on the boundary of Δ, and the second term gives

$$f(0) = -\frac{1}{\pi} \int_\Delta \frac{1}{z} \frac{\partial f}{\partial\overline{z}} \operatorname{Vol} = \frac{1}{\pi} \int_\Delta f \frac{\partial z^{-1}}{\partial\overline{z}} \operatorname{Vol}. \quad (**)$$

by Stokes' theorem. Since

$$\overline{\partial}\partial\log(|z|^2) = \overline{\partial}(z^{-1}dz) = \frac{\partial z^{-1}}{\partial\overline{z}} d\overline{z}\wedge dz,$$

(**) immediately brings $f(0) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta f(z)\partial\overline{\partial}\log|z|$. ■

Poincaré-Lelong formula

The main result today:

THEOREM: (Poincaré-Lelong formula)
Let f be a holomorphic function on a complex manifold M. Then $dd^c \log|f|$ is plurisubharmonic. If, in addition, 0 is a regular value of f, then $\frac{1}{2\pi\sqrt{-1}} dd^c \log|f|$ **is equal to the integration current of the zero divisor of f.**

Proof: Let $l(z) := \log|z|$. Clearly, $f^*(l) = \log|f|$. Then

$$\langle dd^c \log|f|, \tau\rangle = \langle \log|f|, dd^c\tau\rangle = \langle f^*(l), dd^c\tau\rangle = \langle l, dd^c f_*\tau\rangle = \\ = \langle dd^c l, f_*\tau\rangle = \langle \delta_0, f_*\tau\rangle = \frac{1}{2\pi\sqrt{-1}} f_*\tau(0),$$

with $f_*\tau(0)$ being the integral of τ over $f^{-1}(0)$, because the pushforward of a form is its fiberwise integral. ■

COROLLARY: Let $f_1, ..., f_n$ be a collection of holomorphic functions on a complex manifold. **Then $\log\left(\sum_i |f_i|^2\right)$ is plurisubharmonic.**

Proof: Let $u_i := \log|f_i|^2$; by Poincaré-Lelong, this function is plurisubharmonic. Then $\log\left(\sum_i |f_i|^2\right) = \log\left(\sum_i e^{u_i}\right)$ is also plurisubharmonic (Lecture 16). ■

The rest of this lecture **omits most proofs;** *proofs can be found in books by J.-P. Demailly,* `https://www-fourier.ujf-grenoble.fr/~demailly/documents.html`*: "Complex analytic and differential geometry" and "Analytic Methods in Algebraic Geometry"*

Regularized maximum

DEFINITION: (Demailly)
Let $\mu : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth, convex function, monotonous in both arguments. Suppose that for all $|x-y| \geqslant \varepsilon$, one has $\mu(x,y) = \max(x,y)$, and also $\mu(x,y) = \mu(y,x)$, $\mu(y+\alpha, x+\alpha) = \mu(x,y)$. Then μ is called **a regularized maximum** and denoted as $\max_\varepsilon(x,y)$.

REMARK: A regularized maximum of smooth plurisubharmonic functions is smooth and psh.

DEFINITION: A nef current is a positive current obtained as a weak limit of closed, positive forms.

EXAMPLE: Let $x \in M$ be a point on a Kähler manifold, and dist_x the corresponding distance function. It is easy to see that **around x, $dd^c \log \operatorname{dist}_x$ is plurisubharmonic.** Since

$$\log \operatorname{dist}_x = \lim_{C \longrightarrow -\infty} \max_\varepsilon(\log \operatorname{dist}_x, C),$$

$dd^c \log \operatorname{dist}_x$ is a nef current, in a neighbourhood of x where it is positive.

Lelong sets

DEFINITION: Let α be a positive, closed current, and η a nef current, $\eta = \lim \eta_i$, with η_i smooth, positive and closed. Define the product $\alpha \wedge \eta := \lim \alpha \wedge \eta_i$. **This limit exists by compactness, it is closed and positive.**

A caution: The limit **may be non-unique.**
DEFINITION: Choose $\eta = dd^c \log \operatorname{dist}_x$ and η_i its approximation constructed using the regularized maximum. For a closed, positive (p,p)-current Θ, define **the Lelong number $\nu_x(\Theta)$** as a mass of a measure $\Theta \wedge (dd^c \log \operatorname{dist}_x)^{n-p} := \lim_i (\Theta \wedge \eta_i)^{n-p}$ carried at x.
REMARK: In this particular case, the limit $\lim_i \Theta \wedge \eta_i^{n-p}$ **is actually unique**, as follows from the Chern-Levine-Nirenberg inequality.

DEFINITION: Let $s \in \mathbb{R}^{>0}$. The **Lelong set** Z_s of η is the set of all $x \in M$ such that $\nu_x(\Theta) \geqslant s$.

THEOREM: For any current Θ and any $s > 0$, **The Lelong set Z_s is complex analytic (Y.-T. Siu, 1974).**

Siu's decomposition formula: Let Θ be a positive (p,p)-current, and Z_i the p-dimensional components of its Lelong sets, with Lelong numbers c_i (at generic point). **Then $\Theta = \sum_i c_i [Z_i] + R$, where R is closed, positive, and all Lelong sets or R are less than p-dimensional.**

The multiplier ideals

DEFINITION: Let f be a real locally integrable function on a complex manifold, such that $dd^c f + \alpha$ is a positive current, for some smooth $(1,1)$-form α. Then f is called **almost plurisubharmonic**.

DEFINITION: Let L be a line bundle and h a smooth Hermitian metric on L. For any almost plurisubharmonic function f, we call he^{-f} **a singular metric** on L. Its curvature is equal to $\Theta_h + dd^c f$.

DEFINITION: Let f be an almost plurisubharmonic function, and e^{-f} the corresponding singular metric on a trivial line bundle $\mathcal{O}_M$. **The multiplier ideal** of f a sheaf of L^2-integrable holomorphic sections of $\mathcal{O}_M$.

THEOREM: (Nadel) **It is a coherent sheaf.**

REMARK: The multiplier ideal of f **is determined uniquely by the corresponding current $dd^c f$.**

REMARK: $e^{-2\varphi}$ **is integrable in x if and only if the multiplier ideal of φ is trivial in x.**

REMARK: Supports of multiplier ideals I_λ for $e^{-\lambda f}$ **belong to the Lelong sets Z_c of the current $dd^c f$ fpr appopriate $c > 0$.** Conversely, **every Z_c belongs to the support of I_λ for λ sufficiently big.**

Demailly's regularization theorem

DEFINITION: Let η be a closed (1,1)-current on a complex manifold M. We say that η **has algebraic singularities** if for every point $x \in M$ there is a neighbourhood $U \ni x$ and a collection of holomorphic functions $f_1, ..., f_n \in H^0(\mathcal{O}_U)$ such that $\eta = dd^c \log\left(\sum_i |f_i|^2\right) + \eta_0$, where η_0 is smooth.

THEOREM: (The Demailly's Regularization Theorem)
Let T be a positive, closed (1,1)-current on a compact complex Hermitian manifold (M,ω). Then **T is a limit of a sequence $\{T_k\}$ of closed (1,1)-currents with algebraic singularities in the same cohomology class,** such that $T_k + \varepsilon_k\omega$ is positive, and $\lim_k \varepsilon_k = 0$. Moreover, **for each $x \in M$, the Lelong numbers $\nu_x(T_k)$ converge to $\nu_x(T)$ monotonously.**

COROLLARY: A current T with zero Lelong numbers is nef.
Proof: By regularization theorem, $T = \lim_k T_k + \varepsilon_k\omega$, but the singular set of each T_k is empty, because currents with non-trivial algebraic singularities have positive Lelong numbers. **Therefore, each current $T_k + \varepsilon_k\omega$ is smooth.** ■