Complex surfaces

lecture 18: Manifolds of Fujiki class C

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Bimeromorphic map

DEFINITION: Meromorphic map $\varphi : Z \dashrightarrow Z_1$ of complex analytic varieties is a map, defined outside of a nowhere dense subvariety, which in local coordinates can be expressed by meromorphic functions.

DEFINITION: A meromorphic map $\varphi : X \dashrightarrow Y$ is called **bimeromorphic** if there exists a meromorphic map $\psi : Y \dashrightarrow X$ such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are identities in each point where these compositions are defined.

CLAIM: A bimeromorphic map $\varphi : X \dashrightarrow Y$ **induces an isomorphism between a Zariski open subset of X and a Zariski open subset of Y.**

REMARK: Equivalently, **one could define a bimeromorphic map as a subvariety $Z \subset X \times Y$ which projects to X and Y properly and bijectively outside of a Zariski closed set. These definitions are equivalent.**

Weak factorization theorem

THEOREM: (Abramovich, Karu, Matsuki, Włodarczuk)
Let $\varphi : X_1 \dashrightarrow X$ be a bimeromorphic map. **Then φ can be decomposed onto a composition of blow-ups and blow-downs** with smooth center.

Proof: see `https://arxiv.org/pdf/math/0002084` (Kenji Matsuki, Lectures on Factorization of Birational Maps) ■

REMARK: Generally speaking, we take blow-ups, then blow-downs, then blow-ups again and so on. It is conjectured **("strong factorization conjecture")** that we could decompose φ onto a composition of a few blow-ups and then a few blow-downs, but **this conjecture is still open.**

For surfaces, **the strong factorization conjecture was proven by Zariski.**

THEOREM: (Zariski, 1931)
Let $\varphi : X_1 \dashrightarrow X$ be a bimeromorphic map of complex surfaces. Then one can decompose φ onto a composition of a few blow-ups and then a few blow-downs (in this order). More precisely, **φ can be decomposed onto a composition of blow-ups of (smooth) points and blow-downs of smooth rational curves with self-intersection (-1).**

Manifolds of Fujiki class C

DEFINITION: A compact complex manifold is called **Fujiki class C** if it is bimeromorphic to a Kähler manifold.

DEFINITION: Let M be a compact complex manifold, and ω a Hermitian form. **A Kähler current** is a closed, positive $(1,1)$-current η on M such that $\eta - \varepsilon\omega$ is positive for some $\varepsilon > 0$.

THEOREM: (Demailly, Păun)
A compact complex manifold is Fujiki class C **if and only if it admits a Kähler current.**
REMARK: This theorem is highly non-trivial; **we will not use it.**

Today we prove
THEOREM: A complex surface of Fujiki class C is Kähler.

The proof will follow **if we prove that a blow-up of a Kähler surface is Kähler and the blow-down of a Kähler surface is Kähler.**

We also prove
CLAIM: Any Fujiki class C manifold **admits a Kähler current.**

THEOREM: Any complex surface admitting a Kähler current **is Kähler**.

Blow-up of a Kähler surface

PROPOSITION: A blow-up of a Kähler manifold in a point is Kähler.

Proof. Step 1: Let $x \in (M,\omega)$ be a point, and $\pi: \tilde{M} \to M$ a blow-up of M in x. Choose an appropriate coordinate system in a neighbourhood U of x and a cut-off function ψ with support in a small open ball $B_\varepsilon(x)$, equal to 1 in $B_\varepsilon(x)$. Denote by l the logarithm of the Euclidean distance to x; **the function ψl cal be smoothly extended to M, because it is supported in $B_{2\varepsilon}(x)$.**

Step 2: For a sufficiently small ε, the current $dd^c(\varepsilon\psi l) + \omega$ is positive. Indeed, it is positive on $B_\varepsilon(x)$, because $dd^c l = dd^c(\psi l)$ is positive there. it is positive outside of $B_{2\varepsilon}(x)$, because $\psi = 0$ there, and on the annulus $B_{2\varepsilon}(x) \backslash B_\varepsilon(x)$, the form $dd^c(\psi l)$ is bounded, hence satisfies $\omega + \varepsilon dd^c(\psi l)$ for ε sufficiently small.

Step 3 Let $E \subset \tilde{M}$ be the exceptional set of the blow-up; it is isomorphic to $\mathbb{C}P^{n-1}$. On the blow-up, $dd^c(\varepsilon\psi l) + \omega$ is Kähler outside of E (Step 2) and **smooth and Kähler on E by Excerise 4.7 (Assignment 4).** ■

Pushforward of a Kähler form

CLAIM: Let $\psi : (M_1,\omega_1) \longrightarrow (M,\omega)$ be a proper surjective bimeromorphic holomorphic map of complex Hermitian n-manifolds. **Then $\psi_*\psi^*\alpha = \alpha$ for any smooth form $\alpha \in \Lambda^*(M)$.**

Proof: By definition, $\langle \psi_*\psi^*\alpha, \tau\rangle = \langle \psi^*\alpha, \psi^*\tau\rangle = \int_{\tilde M} \psi^*\alpha \wedge \psi^*\tau = \int_M \alpha\wedge\tau = \langle\alpha,\tau\rangle$, because ψ is bijective outside of a measure 0 set. ■

THEOREM: Let $\tilde M \longrightarrow M$ be a holomorphic, birational, proper map of complex manifolds. Then **a pushforward of a Kähler form is a Kähler current**

Proof: Clearly, $\tilde\omega \geqslant \pi^*\omega$ for any Hermitian form ω on M such that

$$\pi : (\tilde M, \tilde\omega) \longrightarrow (M,\omega)$$

is Lipschitz. Since $\|D\pi\|$ is bounded, it is C-Lipschitz for C sufficiently big; rescaling ω, we may always assume that π is Lipschitz. Since $\tilde\omega \geqslant \pi^*\omega$, we have also $\pi_*\tilde\omega \geqslant \pi_*\pi^*\omega = \omega$. This implies that **pushforward of a Kähler form is a Kähler current**. ■

REMARK: The same argument proves that **a pushforward of a Kähler form is a Kähler current,** for any proper surjective holomorphic map of complex n-manifolds (left as an exercise).

Blow-down of a Kähler surface

PROPOSITION: A blow-down of a Kähler surface is Kähler. In other words, consider a blow-up map $\pi : \tilde{M} \to M$ where $(\tilde{M}, \tilde{\omega})$ is a Kähler surface; **then M is also Kähler.**

Proof. Step 1: Since M is a surface, $\pi : \tilde{M} \to M$ is a composition of blow-ups. Therefore, it would suffice to show that M is Kähler when π is a blow-up of M in $x \in M$. The pushforward current is smooth and satisfies $\pi_* \tilde{\omega} \geqslant \omega$ outside of x, but it is singular in x. Let $E \subset \tilde{M}$ be the blow-up curve. Using the local dd^c-lemma, we can assume that $\pi_* \tilde{\omega} = dd^c f$ in some neighbourhood $U \ni x$. **The function f satisfies $f(x) = -\infty$,** because otherwise $\pi^* f$ restricted to E is bounded; however, $dd^c \pi^* f = \tilde{\omega}$, which is impossible, because $\int_E \tilde{\omega} > 0$.

Step 2: Let g be a a potential of a Kähler form on U; it always exists, if U is an open ball. Then $\max_\varepsilon(g - C, f)$ is equal to $g - C$ in a small neighbourhood of x, where $f \leqslant -C$, and is equal to f in a set $U_1 \subset U$, obtained from U by removing a small neighbourhood of x. **Then $dd^c \max_\varepsilon(g - C, f)$ is a Kähler form on U which is equal to $\pi_* \tilde{\omega}$ outside of U_1;** we extend it to $\pi_* \tilde{\omega}$ on $M \backslash U_1$, and obtain a Kähler form on M. ■

Regularization of positive currents

DEFINITION: Let η be a closed (1,1)-current on a complex manifold M. We say that η **has algebraic singularities** if for every point $x \in M$ there is a neighbourhood $U \ni x$ and a collection of holomorphic functions $f_1, ..., f_n \in H^0(\mathcal{O}_U)$ such that $\eta = dd^c \log\left(\sum_i |f_i|^2\right) + \eta_0$, where η_0 is smooth.

THEOREM: (The Demailly's Regularization Theorem)
Let T be a positive, closed (1,1)-current on a compact complex Hermitian manifold (M, ω), which satisfies $T \geqslant \gamma$ for some smooth (1,1)-form γ. Then **T is a limit of a sequence $\{T_k\}$ of closed (1,1)-currents with algebraic singularities in the same cohomology class,** such that $T_k + \varepsilon_k \omega$, is positive, $T_k \geqslant \gamma - \varepsilon_k \omega$, and $\lim_k \varepsilon_k = 0$. Moreover, **for each $x \in M$, the Lelong numbers $\nu_x(T_k)$ converge to $\nu_x(T)$ monotonously.**

THEOREM: (Siu decomposition theorem)
Let Θ be a positive $(1,1)$-current on a complex n-manifold, and Z_i the $(n-1)$-dimensional components of its Lelong sets, with Lelong numbers c_i (at generic point). **Then $\Theta = \sum_i c_i [Z_i] + R$, where R is closed, positive, and all Lelong sets of R have dimension $\leqslant n-2$.**

REMARK: For surfaces, this means that for any Kähler current η on a surface, **$\eta = \eta_0 + \sum_i c_i [Z_i]$, where η is a Kähler current with all Lelong sets 0-dimensional, and $Z_i \subset M$ complex curves.**

Kähler current on a complex surface

REMARK: When we say "singularities" of a current which is locally given by $dd^c f$, where f is smooth outside of the set S where it is equal to $-\infty$, **the "singular set" of this current is S.**

THEOREM: Let M be a complex surface admitting a Kähler current T. **Then M is Kähler.**

Proof. Step 1: Using Demailly regularization, we may replace T by T_k which has algebraic singularities. Indeed, $T \geqslant \gamma$, where γ is Hermitian, and $T_k \geqslant \gamma - \varepsilon_k \omega$, with ε_k converging to 0, hence T_k is Kähler for $k \gg 0$. From Siu decomposition, we may also assume that all singularities of T_k are 0-dimensional. Finally, there are finitely many such singularities, because for any current $dd^c \log\left(\sum_i |f_i|^2\right) + \eta_0$, with isolated algebraic singularities, its Lelong set is the set of common zeros of f_i, which is finite, because it is complex analytic. **It remains to prove the theorem when T is a Kähler current with isolated singularities.**

Step 2: Let $z_1, ..., z_n$ be these singularities, and $dd^c f_i = T$ its potential in a small neighbourhood V_i of each z_i. Then $f_i = -\infty$ in z_i. Choose a Kähler potential ψ_i for a smooth Kähler form in each V_i, and replace f_i by $f_i' := \max_\varepsilon(f_i, \psi_i - C)$, $C \gg 0$. Then $dd^c f_i'$ is smooth, Kähler, and equal to T outside of a small neighbourhood W_i of $\cup_i \{z_i\}$. Gluing $dd^c f_i'$ inside W_i to T outside if W_i, **we obtain a Kähler form on M.** ■