Complex surfaces

lecture 19: Bott-Chern classes represented by Kähler currents

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IMPA, sala 236

February 19,21,24, 2024, 17:00

Kähler currents (reminder)

DEFINITION: A compact complex manifold is called **Fujiki class C** if it is bimeromorphic to a Kähler manifold.

DEFINITION: Let M be a compact complex manifold, and ω a Hermitian form. **A Kähler current** is a closed, positive $(1,1)$-current η on M such that $\eta - \varepsilon\omega$ is positive for some $\varepsilon > 0$.

THEOREM: (Demailly, Păun)
A compact complex manifold is Fujiki class C **if and only if it admits a Kähler current.**
REMARK: This theorem is highly non-trivial; **we will not use it.**

In Lecture 18 we proved
THEOREM: Any complex surface admitting a Kähler current **is Kähler**.

Cohomology classes represented by Kähler currents

REMARK: Let ω be a Gauduchon metric on a complex n-manifold, and η a closed (1,1)-form. We define **the degree** $\deg_\omega \eta := \int_M \eta \wedge \omega^{n-1}$. Since $\int_M dd^c f \wedge \omega^{n-1} = 0$, this number depends only on the cohomology class $[\eta] \in H^{1,1}_{BC}(M, \mathbb{R})$.

THEOREM: (Ahcène Lamari, Lecture 14)
Let M be a compact complex n-manifold. A non-zero class $\eta \in H^{1,1}_{BC}(M, \mathbb{R})$ **can be represented by a positive, closed current if and only if $\deg_\omega \eta > 0$ for any Gauduchon metric ω.**

This lecture we prove another theorem, also due to Lamari.
THEOREM: (Ahcène Lamari)
Let (M, ω) be a compact Hermitian complex manifold. A non-zero class $\eta \in H^{1,1}_{BC}(M, \mathbb{R})$ **can be represented by a Kähler current if and only if there exists $\varepsilon > 0$ such that $\int_M \omega_1^{n-1} \wedge \eta > \varepsilon \int_M \omega_1^{n-1} \wedge \omega$ for any Gauduchon metric ω_1.**
REMARK: The number $\int_M \omega_1^{n-1} \wedge \eta = \deg_{\omega_1} \eta$ is a cohomological invariant of the Bott-Chern class of η and the Aeppli class of ω_1^{n-1}. **The number $\int_M \omega_1^{n-1} \wedge \omega$ has no cohomological meaning,** it depends on the choice of ω_1^{n-1} in its Aeppli cohomology class. **The inequality $\int_M \omega_1^{n-1} \wedge \eta > \varepsilon \int_M \omega_1^{n-1} \wedge \omega$ is given by continuously many conditions**, enumerated by the Gauduchon forms.

Open subsets in currents

REMARK: The topology on currents is "Tychonoff-type", it is the weak topology, with a base of open sets obtained as finite intersections of $U_\tau(]a,b[) := \{x \in D^*(M) \ \mid \ \langle x, \tau\rangle \in]a,b[\}$. This is why (for instance) **the cone of Kähler currents is not open.**

DEFINITION: Let (M,ω) be a compact Hermitian complex n-manifold, and $\varepsilon < 1$ a positive real number. Define a neighbourhood of zero $B_\varepsilon \subset D^{n-1,n-1}_{\mathbb R}(M)$ consisting of all Ξ which satisfy $\left|\int_M \Xi \wedge \omega\right| < \varepsilon$. Let $\operatorname{Pos} \subset D^{n-1,n-1}_{\mathbb R}(M)$ be the cone of positive $(n-1,n-1)$-currents. For any $x \in \operatorname{Pos}$, consider the open ball $B(x) = x + B_\varepsilon$, where $\varepsilon = \int_M x \wedge \omega$. **Define** $\operatorname{Pos}_\varepsilon(\omega) := \bigcup_{x\in \operatorname{Pos}} B_x$. Clearly, $\operatorname{Pos}_\varepsilon$ is open.

CLAIM: The set $\operatorname{Pos}_\varepsilon(\omega)$ is convex.

Proof: For any $x, y \in \operatorname{Pos}$, the set $B(x+y)$ contains $B(x) + B(y)$ because $\int_M (x+y) \wedge \omega = \int_M x \wedge \omega + \int_M y \wedge \omega$. ■

CLAIM: The intersection $\bigcap_\varepsilon \bigcap_\omega \operatorname{Pos}_\varepsilon(\omega)$ over all $\varepsilon \in]0,1[$ and all Hermitian ω **is equal to the positive cone.**

Proof: Clearly, positive cone is the set of all currents η such that $\langle \eta, \omega\rangle \geqslant 0$ for all ω which are Hermitian. Since $\eta \in \operatorname{Pos}_\varepsilon(\omega)$ implies $\langle \eta, \omega\rangle \geqslant 0$, this means that all elements of $\bigcap_\varepsilon \bigcap_\omega \operatorname{Pos}_\varepsilon(\omega)$ satisfy $\langle \eta, \omega\rangle \geqslant 0$ for all Hermitian ω. ■

Countable, dense sets of Gauduchon forms

LEMMA: The space of all Gauduchon forms on a compact complex n-manifold M **has countable, dense subset.**

Proof. Step 1: Gauduchon theorem provides a natural continuoups map Ψ from the set of all Hermitian forms on M to the set of all Gauduchon forms (to fix the ambiguity with the constant multiplier, we choose $\Psi(\omega)$ in such a way that $\int_M \omega^n = \int_M \Psi(\omega)^n$. Therefore, it would **suffice to show that the space of all Hermitian forms has a dense, countable subset.**

Step 2: Since Hermitian forms are open in the space of all smooth (1,1)-forms, it suffices to show that the space $\Lambda^{1,1}(M,\mathbb{R})$ of (1,1)-forms has a dense countable subset. Since open sets in $\Lambda^{1,1}(M,\mathbb{R})$ are open sets in one of the C^i-topologies, it would suffice to show that $(\Lambda^{1,1}(M,\mathbb{R}), C^i)$ has a dense, countable subset. **We leave this as an exercise to discuss in class.** ∎

Compactness of positive currents with bounded mass

REMARK: Let ω be a Hermitian form on a compact complex n-manifold. Recall that **the mass** of a positive current η is $M(\eta) := \int_M \eta \wedge \omega^{n-1}$.

PROPOSITION: The set of all positive $(1,1)$-currents η with $M(\eta) \leqslant C$ **is compact.**

Proof: Positive currents are positive (1,1)-forms with measure coefficients; the condition $M(\eta) \leqslant C$ means that these measures are bounded by a constant, and the space of such measures is compact by Banach-Alaoglu theorem. ■

Aeppli cohomology (reminder)

DEFINITION: Let M be a complex manifold, and $H^{p,q}_{AE}(M)$ the space of dd^c-closed (p,q)-forms modulo $\partial(\Lambda^{p-1,q}M) + \overline{\partial}(\Lambda^{p,q-1}M)$. Then $H^{p,q}_{AE}(M)$ is called **the Aeppli cohomology** of M.

THEOREM: (A. Aeppli)
Let M be a compact complex n-manifold. Then **the Aeppli cohomology is finite-dimensional.** Moreover, the natural pairing

$$H^{p,q}_{BC}(M) \times H^{n-p,n-q}_{AE}(M) \longrightarrow H^{2n}(M) = \mathbb{C},$$

taking x, y to $\int_M x \wedge y$ **is non-degenerate and identifies $H^{p,q}_{BC}(M)$ with the dual $H^{n-p,n-q}_{AE}(M)^*$.**

Proof: Use the same argument as used to prove Serre's duality and Poincaré duality. ■

DEFINITION: Let M be a compact complex n-manifold. Its **Gauduchon cone** is the set of all Aeppli classes of ω^{n-1}, where ω is a Gauduchon form.

REMARK: A $(n-1, n-1)$-current is Aeppli cohomologous to 0 **if and only if it is $(n-1, n-1)$-part of an exact current.**

Intersection of all $\operatorname{Pos}_\varepsilon(\omega)$ with Gauduchon ω

Proposition 1: The intersection $\bigcap_\varepsilon \bigcap_\omega \operatorname{Pos}_\varepsilon(\omega)$ over all $\varepsilon \in]0,1[$ and all Gauduchon ω **is equal to** $\operatorname{Pos}^{1,1} + dd^c D^0_{\mathbb{R}}(M)$, where $\operatorname{Pos}^{1,1}$ is the cone of positive (1,1)-currents.

Proof. Step 1: Clearly, $\operatorname{Pos}^{1,1} + dd^c D^0_{\mathbb{R}}(M)$ paired with a Gauduchon form is positive. **This implies that** $\operatorname{Pos}^{1,1} + dd^c D^0_{\mathbb{R}}(M) \subset \bigcap_\varepsilon \bigcap_\omega \operatorname{Pos}_\varepsilon(\omega)$**.**

Step 2: Conversely, let $\{\omega_\alpha\}$ be a dense countable set of Gauduchon forms. We order the set $\{\omega_\alpha\} \times \{2^{-1}, 2^{-2}, ...\}$ **obtaining a sequence** $\{\operatorname{Pos}_{\varepsilon_i}(\omega_i)\}$ **such that** $\bigcap_i \operatorname{Pos}_{\varepsilon_i}(\omega_i) = \bigcap_\varepsilon \bigcap_{\textbf{all Gauduchon } \omega} \operatorname{Pos}_\varepsilon(\omega)$**, and** $\lim_i \varepsilon_i = 0$**.** Note that for each ω_i, there are many instances of $\operatorname{Pos}_{2^{-k}}(\omega_i)$ in $\{\operatorname{Pos}_{\varepsilon_i}(\omega_i)\}$, for countably many different k.

Step 3: The set $\operatorname{Pos}_{\varepsilon_i}(\omega_i)$ is a union of all $a+b$, where a is positive, and b satisfies the inequality $|\int_M b \wedge \omega_i^{n-1}| \leqslant \varepsilon_i \int_M b \wedge \omega_i^{n-1}$. Then $\bigcap_{i=1}^k \operatorname{Pos}_{\varepsilon_i}(\omega_i)$ is the set of all currents of form $a+b$, **where a is positive, and b satisfies the inequality $|\int_M b \wedge \omega_i^{n-1}| \leqslant \varepsilon_i \int_M b \wedge \omega_i^{n-1}$ for all $i = 1, ..., k$.**

Step 4: If $\eta \in \bigcap_{i=1}^k \operatorname{Pos}_{\varepsilon_i}(\omega_i)$, this means that $\eta = a_k + b_k$, where a_k is positive and $|\int_M b_k \wedge \omega_i^{n-1}| \leqslant \varepsilon_i \int_M a_k \wedge \omega_i^{n-1}$ for $\omega_i = \omega_1, ..., \omega_k$. This condition implies, in particular, that

$$\int_M \eta \wedge \omega_1^{n-1} = \int_M (a_n + b_n) \wedge \omega_1^{n-1} \geqslant (1-\varepsilon_1) \int_M a_n \wedge \omega_1^{n-1}.$$

The set of such a_k is compact, because a_k is a positive current with bounded mass.

Intersection of all $\operatorname{Pos}_\varepsilon(\omega)$ with Gauduchon ω

Proposition 1: The intersection $A := \bigcap_\varepsilon \bigcap_\omega \operatorname{Pos}_\varepsilon(\omega)$ over all $\varepsilon \in]0,1[$ and all Gauduchon ω **is equal to $\operatorname{Pos}^{1,1} + dd^c D^0_{\mathbb{R}}(M)$,** where $\operatorname{Pos}^{1,1}$ is the cone of positive (1,1)-currents.

Steps 2-4: We proved that **any element $\eta \in \bigcap_{i=1}^k \operatorname{Pos}_{\varepsilon_i}(\omega_i)$ satisfies $\eta = a_k + b_k$, where a_k is positive and $|\int_M b_k \wedge \omega_i^{n-1}| \leqslant \varepsilon_i \int_M a_k \wedge \omega_i^{n-1}$ for $\omega_i = \omega_1, ..., \omega_k$;** also, $\{a_k\}$ belongs to a compact set of currents.

Step 5: Replace $\{a_i\}$ by a converging subsequence, and let $a := \lim_i a_i$. Since $\lim_i |\int_M (\eta - a_i) \wedge \omega_k^{n-1}| = 0$ for all k, and $\{\omega_k\}$ are dense in the set of all Gauduchon forms, **this implies that $\int_M (\eta - a) \wedge \omega^{n-1} = 0$ for all Gauduchon ω.**

Step 6: It **remains to show that any real (1,1)-current b which satisfies**

$$\int_M b \wedge \omega^{n-1} = 0 \text{ for all Gauduchon } \omega \quad (*)$$

belongs to $dd^c D^0_{\mathbb{R}} M$. However, Gauduchon forms generate the space of all dd^c-closed forms, which contains the set of all $(n-1, n-1)$-parts of exact forms. Therefore $\langle b, \operatorname{im} d\rangle = 0$, hence $db = 0$. Since $\langle b, u\rangle = 0$ for any dd^c-closed u, this implies that the Bott-Chern class of b, paired with any Aeppli $(n-1, n-1)$-class, vanishes. **By duality between Aeppli and Bott-Chern classes, this implies that b is Bott-Chern exact**, that is, $b \in dd^c D^0_{\mathbb{R}} M$ ∎

Weak dual spaces

DEFINITION: Let X be a topological vector space, and X^* the space of all continuous linear functionals. The **weak topology** on X^* is the weakest on X^* such that for all $x \in X$, the map $\lambda \longrightarrow \lambda(x)$ is continuous.

LEMMA: Let $(X^*_w)^*$ be the dual space to X^*, where X^*_w is taken with the weak topology. **Then the natural map $X \longrightarrow (X^*_w)^*$ is bijective.**

Proof. Step 1: The topology on V^*_w is defined by a system of seminorms $\nu_x(\lambda) = |\lambda(x)|$. Therefore, the continuity of a linear functional ζ on X^*_w means that it satisfies the inequality $|\zeta| \leqslant \sum_i |\nu_{x_i}|$, for a finite collection of points $x_1, ..., x_n \in V$. **This implies that ζ vanishes on the space $W := \langle x_1, x_2, ...\rangle^\perp \subset V^*$, of functionals which vanish on $x_1, ..., x_n$.**

Step 2: Clearly, $V^*/W = \langle x_1, x_2, ...\rangle^*$. Since a finite-dimensional space satisfies $L^{**} = L$, this implies that a natural map $\langle x_1, x_2, ...\rangle \xrightarrow{\Psi} (V^*/W)^*$ is an isomorphism. **Therefore, $\zeta \in \operatorname{im} \Psi$.** ■

Hahn-Banach theorem (reminder)

DEFINITION: We say that **a hyperplane** in a topological vector space V is a closed codimension 1 subspace $H \subset V$.

THEOREM: (Hahn-Banach separation theorem)
Let V be a locally convex topological vector space, $A \subset V$ an open convex subset, and $W \subset V$ a closed subspace. Assume that $W \cap A = \emptyset$. **Then there exists a continuous functional $\xi \in V^*$ such that $\xi(W) = 0$ and $\xi(A) > 0$.**

Gauduchon positive forms

DEFINITION: Let (M,ω) be a compact complex Hermitian n-manifold. A (1,1)-form α is **Gauduchon positive** if there exists a number $\varepsilon > 0$ such that for any positive, dd^c-closed $\psi \in \Lambda_{\mathbb{R}}^{n-1,n-1}(M)$, one has

$$\int_M \alpha \wedge \psi > \varepsilon \int_M \omega \wedge \psi. \quad (***)$$

REMARK: Clearly, the set $A \subset \Lambda_{\mathbb{R}}^{1,1}(M)$ of all Gauduchon positive forms is a convex cone.

REMARK: Later today we will prove that **a Bott-Chern class $[v] \in H_{BC}^{1,1}(M,\mathbb{R})$ contains a Kähler current if and only if it contains a Gauduchon positive closed (1,1)-form.**

The open cone of Gauduchon positive forms

PROPOSITION: Let (M,ω) be a compact complex manifold equipped with a Hermitian metric ω, and $A\subset\Lambda^{1,1}_{\mathbb R}(M)$ the set of Gauduchon positive forms. **Then A is a cone, open in C^0-topology.**

Proof. Step 1: The set A is closed under addition and multiplication by a positive constant. Indeed, if α satisfies (***) with constant ε, then $A\alpha$ satisfies (***) with constant $A\varepsilon$; also, if α_1,α_2 satisfy (***) with constant $\varepsilon_1,\varepsilon_2$, then $\alpha_1+\alpha_2$ satisfies (***) with constant $\varepsilon_1+\varepsilon_2$.

Step 2: Let g be the Riemannian metric associated with ω, and u a (1,1)-form which satisfies $\sup_M \|u\|_g \leqslant \delta$. Then there exists a g-orthonormal basis such that u is orthogonal in this basis with eigenvalues $|\delta_i|<\delta$. Therefore, for any positive $(n-1,n-1)$-form ρ, we have $|\int_M u\wedge\rho| \leqslant \delta\int_M u\wedge\rho$. This implies that for any Gauduchon positive form η which satisfies (***) with constant ε, we have

$$\int_M(\alpha+u)\wedge\psi > \varepsilon\int_M\omega\wedge\psi - |\int_M u\wedge\psi| \geqslant (\varepsilon-\delta)\int_M\omega\wedge\psi.$$

This implies that $\alpha+u$ is Gauduchon positive for $\delta<\varepsilon$, and therefore A is open. ■

Gauduchon positive forms and positive currents

Proposition 2: Let (M,ω) be a compact complex manifold equipped with a Hermitian metric ω. Let $\beta \in \Lambda^{1,1}_{\mathbb{R}}(M)$ be a Gauduchon positive form. **Then there exists a generalized function $f \in D^0(M)$ such that $\beta + dd^c f$ is a positive current.**

Proof. Step 1: Choose a Hermitian form ω_1, and let $\operatorname{Pos}_\varepsilon(\omega_1) \subset D^{1,1}(M)$ the open cone defined above. If $\beta + dd^c D^0(M) \cap \operatorname{Pos}_\varepsilon(\omega_1) = \varnothing$, by Hahn-Banach theorem there exists a form $\rho \in D^{1,1}(M)^* = \Lambda^{n-1,n-1}_{\mathbb{R}}(M)$, positive on $\operatorname{Pos}_\varepsilon(\omega_1)$, **such that $\int_M (\beta + dd^c f) \wedge \rho = 0$, for any $f \in D^0(M)$.** Positivity on $\operatorname{Pos}_\varepsilon(\omega_1)$ **implies that the form ρ is positive,** because $\operatorname{Pos}_\varepsilon(\omega_1)$ contains the cone of positive currents.

Step 2: Since $\int_M \omega \wedge \rho > 0$, the condition $\int_M \beta \wedge \rho = 0$ contradicts Gauduchon positivity of β. **This implies that $\beta + dd^c D^0(M) \cap \operatorname{Pos}_\varepsilon(\omega_1) \neq \varnothing$ for all ε, ω_1.**

Step 3: Clearly, $\operatorname{Pos}_\varepsilon(\omega_1) = \operatorname{Pos}_\varepsilon(\omega_1) + dd^c D^0(M)$ when ω_1 is Gauduchon. Therefore, **$\beta + dd^c D^0(M)$ is contained in $\bigcap_\varepsilon \bigcap_{\text{all Gauduchon } \omega_1} \operatorname{Pos}_\varepsilon(\omega_1)$.** This intersection is equal to $\operatorname{Pos} + dd^c D^0(M)$ by Proposition 1. ■

Cohomology classes represented by Kähler currents

Proposition 3: Let (M,ω) be a compact complex Hermitian n-manifold, and $[v]\in H^{1,1}_{BC}(M)$. **Then $[v]$ can be represented by a Kähler current if and only if there exists $\varepsilon>0$ such that**

$$\int_M [v]\wedge\psi > \varepsilon\int_M \omega\wedge\psi \quad (**)$$

for any positive, dd^c-closed $(n-1,n-1)$-form ψ on M.

Proof. Step 1: Let v be a Kähler current, $v>\varepsilon\omega$. **Then $\int_M v\wedge\psi\geqslant\varepsilon\int_M\omega\wedge\psi$.**

Step 2: Conversely, let $[v]$ be a cohomology class which satisfies (**), and α a (1,1)-form representing $[v]$. Then $\alpha' := \alpha-\frac{1}{2}\varepsilon\omega$ is Gauduchon positive. By Proposition 2, $\alpha' = a+dd^c f$, where a is a positive current, and f a generalized function. **This implies that $\alpha - dd^c f = a+\frac{1}{2}\varepsilon\omega$, where $a+\frac{1}{2}\varepsilon\omega$ is a Kähler current.** ■

COROLLARY: A compact complex manifold M admits a Kähler current **if and only if there exists a closed, Gauduchon positive (1,1)-form.**

Proof: Any such form satisfies (**), hence it is Bott-Chern cohomologous to a Kähler current. ■

Nef-pluriharmonic currents

DEFINITION: A current $\eta \in D^{n-1,n-1}(M)$ is called **nef-pluriharmonic** if $\eta = \lim_i \omega_i^{n-1}$, where all ω_i are Gauduchon forms.

Lemma 4: Let (M,ω) be a compact complex Hermitian n-manifold. Then **a non-zero current is nef-pluriharmonic if and only if it is strictly positive on all Gauduchon positive forms.**

Proof. Step 1: Let K° be the space of positive dd^c-closed forms $\psi \in \Lambda_{\mathbb{R}}^{n-1,n-1}(M)$, satisfying $\int \omega \wedge \psi = 1$, and K its closure in the space of currents. Since all currents in K are positive and have bounded mass, this space is compact. Clearly, the set K generates the cone of nef-pluriharmonic currents. Consider a Gauduchon positive $(1,1)$-form α. For any $\psi \in K^\circ$, we have

$$\int_M \alpha \wedge \psi > \varepsilon \int_M \omega \wedge \psi = \varepsilon.$$

Then $\alpha|_K > \varepsilon$. This implies that

$$\big\langle \text{Gauduchon positive}, \text{nef-pluriharmonic}\big\rangle > 0.$$

Step 2: Conversely, assume that $\xi \in D_{\mathbb{R}}^{n-1,n-1}(M)$ is a non-zero current which is obtained as a limit of Gauduchon forms, and α is Gauduchon positive. **Then $\langle \xi, \alpha\rangle \geqslant \varepsilon \xi, \omega\rangle$, because the same inequality is true for all Gauduchon forms, and ξ is a limit of Gauduchon forms.** ■

Existence of Kähler currents

THEOREM: Let M be a compact complex n-manifold. Then M admits a Kähler current **if and only if for any non-zero nef-pluriharmonic current, its Aeppli class is non-zero.**

Proof. Step 1: Let Ξ be a Kähler current on M, and β a non-zero nef-pluriharmonic current with $\beta = \lim_i \beta_i$, where $\beta_i = \omega_i^{n-1}$ and all ω_i are Gauduchon. We are going to prove that $[\beta]_{AE} \neq 0$, **which would follow if we prove $\langle[\Xi],[\beta]\rangle \neq 0$,** where $\langle[\Xi],[\beta]\rangle$ denotes the pairing between the Aeppli and Bott-Chern cohomology.

Step 2: Let ω be a Hermitian form on M. By Proposition 3, there exists $\varepsilon > 0$ such that $\int_M [\Xi] \wedge \beta_i \geqslant \varepsilon \int \omega \wedge \beta_i$. **Passing to a limit, we obtain $\int_M [\Xi] \wedge [\beta] \geqslant \varepsilon \int \omega \wedge \beta > 0$, which implies $\langle[\Xi],[\beta]\rangle > 0$.**

Step 3: Conversely, let A be the cone of Gauduchon positive forms. By Proposition 3, to prove that M admits a Kähler current, it suffices to show that $A \cap \ker d \neq 0$. If the intersection is empty, we apply Hahn-Banach and obtain a current $\xi \in D_{\mathbb{R}}^{n-1,n-1}(M)$ which vanishes on closed (1,1)-forms and is strictly positive on Gauduchon positive (1,1)-forms. Lemma 4 implies that ξ is nef-pluriharmonic. Since ξ vanishes on closed (1,1)-forms, its Aeppli class satisfies $\langle[\xi], H^{1,1}_{BC}(M)\rangle = 0$. Since the pairing $\langle H^{n-1,n-1}_{AE}(M), H^{1,1}_{BC}(M)\rangle$ is perfect, we obtain that $[\xi]_{AE} = 0$. ■