Complex surfaces

lecture 20: Kähler currents on complex surfaces with even b_1

Misha Verbitsky

IMPA, sala 236

February 26, 2024, 17:00

Defect (reminder)

Theorem 1, lecture 11: Let M be a compact surface. **Then the kernel of the natural map $P: H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is at most 1-dimensional.**
DEFINITION: The number $\dim \ker P$ is called **the defect** of a surface, denoted $\delta(M)$; by the previous theorem it can be 1 or 0. In the course of the proof of Lamari's theorem, we will show that **the surface is Kähler if and only if $\delta(M) = 1$.**

Proposition 5, Lecture 12: Let M be a complex surface with non-zero defect. **Then $\ker P$ can be generated by a class $d^c[\theta]$, where $\theta \in H^1(M, \mathbb{R})$, and $H^1(M, \mathbb{C}) = \mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \oplus \langle\theta\rangle$.**

REMARK: This implies that a complex surface with $b_1(M)$ even **has vanishing defect**.

THEOREM: (A. Aeppli)
Let M be a compact complex n-manifold. Then **the Aeppli cohomology is finite-dimensional.** Moreover, the natural pairing

$$H^{p,q}_{BC}(M) \times H^{n-p,n-q}_{AE}(M) \longrightarrow H^{2n}(M) = \mathbb{C},$$

taking x, y to $\int_M x \wedge y$ **is non-degenerate and identifies $H^{p,q}_{BC}(M)$ with the dual $H^{n-p,n-q}_{AE}(M)^*$.**

dd^c-lemma on surfaces with vanishing defect.

THEOREM: On a surface with vanishing defect, the map $P: H^{p,q}_{BC}(M) \longrightarrow H^{p+q}(M)$ is injective, in other words, **the dd^c-lemma holds.**

Proof. Step 1: When $p+q=1$, the elements of $H^{p,q}_{BC}(M)$ are represented by holomorphic or antiholomorphic forms, and **the map**

$$\mathcal{H}^{1,0}(M) \oplus \overline{\mathcal{H}^{1,0}(M)} \longrightarrow H^1(M, \mathbb{C})$$

is injective by Claim 1 from lecture 12.

Step 2: When $p+q=4$, fix a Gauduchon metric ω, and consider an elliptic operator $D(f) := dd^c(f\omega)$ from $C^\infty M$ to $\Lambda^{4,4}(M)$. This map vanishes on constants, and all forms V in its image satisfy $\int_M V = 0$; index theorem implies that it is surjective to the set of such forms. **This proves the dd^c-lemma for $p+q=4$.**

Step 3: The map $H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is injective by definition of defect. The spaces $H^{2,0}_{BC}(M)$ and $H^{0,2}_{BC}(M)$ are identified with the spaces of holomorphic and antiholomorphic 2-forms; these are never exact, because $\int_M \alpha \wedge \overline{\alpha} > 0$ for any non-zero holomorphic or antiholomorphic 2-form. **This proves the dd^c-lemma when $p+q=2$.**

dd^c-lemma on surfaces with vanishing defect (2)

THEOREM: On a surface with vanishing defect, the map $P: H^{p,q}_{BC}(M) \longrightarrow H^{p+q}(M)$ is injective, in other words, **the dd^c-lemma holds.**

Step 4: It remains to prove the dd^c-lemma for $H^{1,2}_{BC}(M)$. By duality, the injectivity of $P: H^{1,2}_{BC}(M) \longrightarrow H^3(M)$ is equivalent to the surjectivity of $P^*: H^1(M) \longrightarrow H^{1,0}_{AE}(M)$. Take a class $[\alpha] \in H^{1,0}_{AE}(M)$ represented by a (1,0)-form α. Since $d^c\alpha$ is closed, $\partial\alpha$ is a closed (2,0)-form. If it is non-zero, the integral $\int_M \partial\alpha \wedge \overline{\partial\alpha}$ has to be positive, which is impossible, because the form $\partial\alpha \wedge \overline{\partial\alpha} = d(\alpha \wedge \overline{\partial\alpha})$ is exact. Then $d^c\alpha$ is a closed (1,1)-form; this form is exact, because $d^c\alpha = Id(I^{-1}\alpha) = -d(I^{-1}\alpha)$. This implies $d^c\alpha = dd^c f$, for some $f \in C^\infty M$. Applying the same argument to $I(\alpha)$, we obtain $d\alpha = dd^c g$. **Then the form $\alpha - d^c g = \alpha - \sqrt{-1}\,(\partial g - \overline{\partial} g)$ is closed and Aeppli cohomologous to α,** implying that $P^*: H^1(M) \longrightarrow H^{1,0}_{AE}(M)$ is surjective, and therefore $P: H^{1,2}_{BC}(M) \longrightarrow H^3(M)$ is injective. ■

Hodge decomposition on surfaces with vanishing defect.

Propositon 1: Let M be a complex surface with vanishing defect. **Then any closed real form α on M is cohomologous to a sum of closed $(2,0)$, $(1,1)$ and $(0,2)$-forms.**

Proof. Step 1: The form $d(\alpha^{2,0})$ is (2,1) and exact, hence $\bar\partial(\alpha^{2,0}) = \bar\partial\partial\beta$ for some $(0,1)$-form β (here we use the dd^c-lemma for $H^{2,1}_{BC}$ we have just proven). Then $(\alpha - d\beta)^{2,0}$ is closed. Similarly, $(\alpha - d\bar\beta)^{0,2}$ is also closed.

Step 2: The form $\alpha - d\beta - d\bar\beta$ is closed and has closed (2,0)-part and (0,2)-part. Therefore, its $(2,0)$, $(1,1)$ and $(0,2)$-parts are closed. ■

Aeppli cohomology on manifolds with dd^c-lemma in $H^{1,1}(M)$.

Corollary 1: Let Ξ be a dd^c-closed $(1,1)$-form on a compact complex surface. Assume that the natural map $P: H^{1,1}_{BC}(M) \longrightarrow H^2(M)$ is injective. **Then Ξ is Aeppli cohomologous to a closed $(1,1)$-form.**

Proof: Consider the map $P^*: H^2(M,\mathbb{R}) \longrightarrow H^{1,1}_{AE}(M,\mathbb{R})$, taking a closed 2-form to its (1,1)-part. This map is Poincaré dual to P; since P is injective, **its dual P^* is surjective.** Then Ξ is Aeppli cohomologous to a (1,1)-part of a closed form $\Xi_1 \in \Lambda^2(M,\mathbb{R})$. However, any closed 2-form on M is cohomologous to a sum of closed (2,0), (1,1) and (0,2)-forms, $(\alpha - d\beta - d\overline{\beta})^{2,0} + (\alpha - d\beta - d\overline{\beta})^{1,1}(\alpha - d\beta - d\overline{\beta})^{0,2}$ (Proposition 1). Then Ξ is Aeppli cohomologous to $(\alpha - d\beta - d\overline{\beta})^{1,1}$, which is closed. ■

Nef-pluriharmonic currents

DEFINITION: A current $\eta \in D^{n-1,n-1}(M)$ is called **nef-pluriharmonic** if $\eta = \lim_i \omega_i^{n-1}$, where all ω_i are Gauduchon forms.

THEOREM: Let M be a compact complex n-manifold. Then M admits a Kähler current **if and only if for any non-zero nef-pluriharmonic current, its Aeppli class is non-zero.**

Proof: Lecture 19. ■

We are going to prove that **on a surface M with vanishing defect, any non-zero nef-pluriharmonic current has non-zero Aeppli class**; **this implies that M is Kähler.**

Nef-pluriharmonic currents and their cohomology

PROPOSITION: Let M be a complex surface with vanishing defect, and Ξ a nef-pluriharmonic current, Aeppli cohomologous to 0. **Then $d\Xi = 0$.**

Proof. Step 1: By definition, Ξ can be obtained as a limit of Gauduchon forms, $\Xi = \lim_i \Xi_i$. Choose a space W of smooth, closed (1,1)-forms such that the natural map $W \longrightarrow H^{1,1}_{AE}(M)$ is an isomorphism (Corollary 1). Then we can decompose $\Xi_i = b_i + a_i$, where a_i are Aeppli exact, and $b_i \in W$. Since W is finite-dimensional, and $\lim_i b_i = 0$, **the sequence $\{b_i\}$ converges to zero in any of C^i-topologies.**

Step 2: Let γ_i be 1-forms which satisfy $(d\gamma_i)^{1,1} = a_i$. Let c_i be the (2,0)+(0,2)-part of $d\gamma_i$. Then

$$0 \leqslant \int_M \Xi_i \wedge \Xi_i = \int_M (b_i + d\gamma_i - c_i) \wedge (b_i + d\gamma_i - c_i) =$$

$$= \int_M (b_i + d\gamma_i) \wedge (b_i + d\gamma_i) + \int_M c_i \wedge c_i - 2\int_M c_i \wedge (b_i + d\gamma_i).$$

The last term can be rewritten as $\int_M c_i \wedge (b_i + d\gamma_i) = \int_M c_i \wedge c_i$, because c_i is (2,0)+(0,2)-part of $b_i + d\gamma_i$, hence c_i multiplied by $b_i + d\gamma_i - c_i$ vanishes. **Summarizing and using $db_i = 0$ and integration by parts, we obtain**

$$0 \leqslant \int_M \Xi_i \wedge \Xi_i = \int_M b_i \wedge b_i - \int_M c_i \wedge c_i. \quad (*)$$

Nef-pluriharmonic currents and their cohomology (2)

Step 2: Let γ_i be 1-forms which satisfy $(d\gamma_i)^{1,1} = a_i$. Let c_i be the (2,0)+(0,2)-part of $d\gamma_i$. Then

$$0 \leqslant \int_M \Xi_i \wedge \Xi_i = \int_M (b_i + d\gamma_i - c_i) \wedge (b_i + d\gamma_i - c_i) =$$
$$= \int_M (b_i + d\gamma_i) \wedge (b_i + d\gamma_i) + \int_M c_i \wedge c_i - 2 \int_M c_i \wedge (b_i + d\gamma_i).$$

The last term can be rewritten as $\int_M c_i \wedge (b_i + d\gamma_i) = \int_M c_i \wedge c_i$, because c_i is (2,0)+(0,2)-part of $b_i + d\gamma_i$, hence c_i multiplied by $b_i + d\gamma_i - c_i$ vanishes. **Summarizing and using $db_i = 0$ and integration by parts, we obtain**

$$0 \leqslant \int_M \Xi_i \wedge \Xi_i = \int_M b_i \wedge b_i - \int_M c_i \wedge c_i. \quad (*)$$

Step 3: On real (2,0)+(0,2)-forms, the quadratic form $x \mapsto \int_M x \wedge x$ is positive definite and satisfies $\|x\|_{L^2} = \int_M x \wedge x$. Then (*) gives $\int_M c_i \wedge c_i = \|c_i\|_{L^2} \leqslant \int_M b_i \wedge b_i$. Since $\lim_i \int_M b_i \wedge b_i = 0$, this gives $\lim_i \|c_i\|_{L^2} = 0$, which implies (by Cauchy-Schwarz inequality) that $\lim_i \|c_i\|_{L^1} = 0$. **This implies that $\lim_i c_i = 0$ in the topology of currents, hence the limit $\Xi = \lim_i \Xi_i = \lim_i b_i + d\gamma_i$ is closed.** ■

Surfaces with vanishing defect are Kähler

THEOREM: Let M be an complex surface with vanishing defect, and Ξ a nef-pluriharmonic current, Aeppli cohomologous to 0. **Then $\Xi = 0$.**

Proof. Step 1: By the previous proposition, $d\Xi = 0$. Since Ξ is Aeppli exact, its intersection pairing with any $x \in H^{1,1}_{BC}(M)$ vanishes. However, $H^2(M)$ is a direct sum of $H^{1,1}_{BC}(M)$, the space of holomorphic and antiholomorphic 2-forms (Proposition 1), hence the intersection pairing of Ξ with $H^2(M)$ also vanishes. **By Poincaré duality, this implies that Ξ is d-exact.**

Step 2: From dd^c-lemma we obtain that $\Xi = dd^c f$, where f is a plurisubharmonic function. Then $\int_M \Xi \wedge \omega = \int_M f \wedge dd^c\omega = 0$ for any Gauduchon form ω. **This implies that Ξ is a positive current with zero mass, hence $\Xi = 0$.** ■

COROLLARY: A complex surface M with even b_1 is Kähler.

Proof: In Lecture 12, we proved that defect of M vanishes if and only if $b_1(M)$ is even. By the previous theorem, vanishing of defect implies that all nef-pluriharmonic currents which are Aeppli cohomologous to 0 vanish. In lecture 19, we proved that this implies that M admits a Kähler current. In Lecture 18, we proved that any surface admitting a Kähler current also admits a Kähler metric. ■