Complex surfaces

lecture 21: LCK metrics on Kato surfaces

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Kato manifolds

DEFINITION: Let B be an open ball in $\mathbb{C}^n$, $n > 1$, and $\tilde{B} \xrightarrow{\pi} B$ be a bimeromorphic, holomorphic map, which is an isomorphism outside of a compact subset. Remove a small ball B_0 in $\tilde{B}$ and glue it to the boundary of $\tilde{B}$, extending the complex structure smoothly (and holomorphically) on the resulting manifold, denoted by M. Then M us called **a Kato manifold.**

The main result today:
THEOREM: (M. Brunella) Suppose that M is a Kato manifold obtained from $\tilde{B} \stackrel{\pi}{B}$ with $\tilde{B}$ Kähler. **Then M is LCK.**

Kato manifolds (2)

THEOREM: (Kato)
Let M be a Kato manifold. Then there exists a family M_t of complex manifolds over a punctured disk **such that $M = M_0$ and all M_t, for $t > \varepsilon$, are bimeromorphic to a Hopf manifold.**

Proof. Step 1: Assume that the ball B_0 does not intersect the exceptional set of $\tilde{B} \xrightarrow{\pi} B$. Then the natural bimeromorphism $\tilde{B} \xrightarrow{\pi} B$ is biholomorphic in a neighbourhood of ∂B_0, hence π can be extended to a map from M to a Hopf manifold obtained by gluing the boundary of B_0 and the boundary of B.

Step 2: Moving the center of the ball B_0 and decreasing its radius, we can always ensure that B_0 does not intersect the blow-up divisor. ■

COROLLARY: A Kato surface **is diffeomorphic to a blown-up of a Hopf surface.** ■

CR-manifolds.

Definition: Let M be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I : B \longrightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_C M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then (B, I) is called **a CR-structure on M**.

Example: A complex manifold is CR, with $B = TM$. Indeed, $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ **is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M \subset X$ a realhypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $\operatorname{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, **$M$ is a CR-manifold.**

Definition: **A Frobenius form of a CR-manifold** is the tensor $B \otimes B \longrightarrow TM/B$ mapping X, Y to the $\Pi_{TM/B}([X, Y])$. It is an obstruction to integrability of the foliation given by B.

Pseudoconvex CR-manifolds.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. This means that it is Hermitian. This Hermitian form is called **the Levi form** of a CR-manifold.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Levi form is positive definite everywhere.

Example: Let h be a function on a complex manifold such that $\partial\bar\partial h = \omega$ is a positive definite Hermitian form, and $X = h^{-1}(c)$ its level set. Then the Frobenius form of X is equal to $\omega|_X$. In particular, X **is a strictly pseudoconvex CR-manifold**.

Level set of a plurisubharmonic function

PROPOSITION: Let M be a complex manifold, $\varphi \in C^\infty(M)$ a smooth function, and s a regular value of φ. Consider $S := \varphi^{-1}(s)$ as a CR-manifold, with $B = TS \cap I(TS)$, and let Φ be its Levi form, taking values in

$$TS/B = \frac{\ker d\varphi}{\ker d\varphi \cap I(\ker d\varphi)}.$$

Then $d^c\varphi : TS/B \longrightarrow C^\infty(S)$ trivializes TS/B. Consider the tangent vectors $u, v \in B$. **Then $-d^c\varphi(\Phi(u,v)) = dd^c\varphi(u,v)$.**

Proof: Extend u, v to vector fields $u, v \in B = \ker d\varphi \cap I(\ker d\varphi)$. Then

$$-d^c\varphi(\Phi(u,v)) = -d^c\varphi([u,v]) = dd^c\varphi(u,v) - \operatorname{Lie}_v d^c\varphi(u) + \operatorname{Lie}_u d^c\varphi(v).$$

(the last equality follows from the Cartan formula). However, $d^c\varphi(u) = d^c\varphi(v) = 0$ because $v, u \in I(\ker d\varphi)$. This gives $-d^c\varphi(\Phi(u,v)) = dd^c\varphi(u,v)$. ∎

Stein filling

DEFINITION: Let (M, B, I) be a CR-manifold. A function f on M is called **CR-holomorphic** if for any vector field $v \in B^{0,1}$, we have $\operatorname{Lie}_v f = 0$.

THEOREM: (H. Rossi, A. Andreotti, Y.-T. Siu)
Let S be a compact strictly pseudoconvex CR-manifold, $\dim_{\mathbb R} S \geqslant 5$, and $H^0(\mathcal{O}_S)$ the ring of CR-holomorphic functions. **Then S is the boundary of a Stein variety M with isolated singularities,** such that $H^0(\mathcal{O}_S) = H^0(\mathcal{O}_M)$, where $H^0(\mathcal{O}_M)$ denotes the ring of holomorphic functions on M, considered a compact complex manifold with boundary, and $\mathcal{O}_S$ is the ring of CR-holomorphic functions.

DEFINITION: Let S be a strictly pseudoconvex CR-manifold, obtained as a boundary of a Stein variety M. Then M is called **Stein filling** of S,

CLAIM: The Stein filling is unique in dimension 3, though existence is false in dimension 3, and true in dimension $\geqslant 5$. Also, **holomorphic automorphisms of M are in bijective correspondence with CR-holomorphic automorphisms of S.**

Proof: Stein variety is uniquely determined by its ring of holomorphic functions (Forster). ■

Levi problem

DEFINITION: A **strictly pseudoconvex domain** is an open subset such that its boundary is a strictly pseudoconvex real hypersurface in $\mathbb{C}^n$.

REMARK: "Levi problem", posited by Eugenio Elia Levi (1883–1917), **asked whether any strictly pseudoconvex domain is Stein;** it was solved by Oka for $\mathbb{C}^2$ (1942), and for arbitrary dimension independently in 1953 by Oka, Bremermann and Norguet.

Class VII surfaces (reminder)

DEFINITION: A complex surface is a compact complex manifold M of complex dimension 2. Let M be a compact complex manifold, and K_M its canonical bundle. The **canonical ring** $\oplus_{i=0}^{\infty} H^0(K^i)$ is finitely generated for for all projective varieties (Birkar, Cascini, Hacon, McKernan), for complex surfaces (Kodaira). Conjecturally, it is **always finitely generated.** Let $a \in \mathbb{Z}^{>0}$. Consider the function $P_a(N) = H^0(K^{aN})$. If the canonical ring is finitely generated, the function $N \mapsto P_a(N)$ **is polynomial** for a which divides all degrees of its generators **(prove this).** The degree $\kappa(M)$ of this polynomial is called **the Kodaira dimension** of M. If $H^0(K^i) = 0$ for all $i > 0$, we set $\kappa(M) = -\infty$.

DEFINITION: Class VII surface (also called Kodaira class VII surface) is a complex surface with $\kappa(M) = \infty$ and first Betti bumber $b_1(M) = 1$. Minimal class VII surfaces are called **class VII$_0$ surfaces.**

REMARK: Kodaira defined the "Class VII" in another, non-equivalent way. The current "Class VII" is Kodaira's "class 7" from his version of Kodaira-Enriques classification, published 1966. The term "class VII" with its current meaning is due to Barth, Peters, Van de Ven.

The global spherical shell

DEFINITION: Let $S \subset \mathbb{C}^2$ be a standard sphere, and S_ε its ε-neighbourhood. A complex surface M **admits a global spherical shell** (GSS) if there is a holomorphic embedding $S_\varepsilon \longrightarrow M$, for some $\varepsilon > 0$, such that the complement of its image is connected.

THEOREM: (Ma. Kato)
A complex surface **admits a global spherical shell if and only if it is a Kato surface.**

Proof: It is clear that a Kato surface admits a GSS. Conversely, consider a GSS surface M, let $S \subset M$ be its spherical shell, and M_1 the complex surface with boundary of two copies of S, obtaining by cutting M in S. Fill the interior part of the boundary by a ball. This gives a compact manifold M_1 with its boundary CR-isomorphic to a sphere. The global holomorphic functions on M_1 are the same as CR-holomorphic functions on its boundary, which gives a natural map from M_1 to its Stein filling, which is by construction bimeromorphic. ■

The GSS conjecture

CONJECTURE: (GSS conjecture, due to Ma. Kato)
Let M be a class VII surface with $b_2 > 0$. **Then M is a Kato surface.**

REMARK: A. Teleman **proved the GSS conjecture when $b_2(M) = 1$.**

REMARK: By results of G. Dlousky, K. Oeljeklaus and M. Toma, a Kato surface M admits at least $b_2(M)$ distinct rational curves, and, conversely, **if a complex surface admits $b_2(M)$ distinct rational curves in a certain configuration, it is a Kato surface.**

Brunella's theorem

THEOREM: (Brunella)
Suppose that M is a Kato surface obtained from an iterated blow-up $\hat{B} \xrightarrow{\pi} B$ and an open ball $B_0 \subset \hat{B}$. **Then M is LCK.**

Step 1. Strategy of the proof: Let $\Psi: B \longrightarrow B_0$ be the biholomorphism used to glue two components of the boundary of $B \backslash B_0$. We find a Kähler metric $\hat{\omega}$ on $\hat{B}$ with the following automorphic condition. Consider the space $\tilde{M}$ obtained by gluing $\mathbb{Z}$ copies of $\hat{B} \backslash B_0 = M_i, i \in \mathbb{Z}$ as above. A Kähler metric $\tilde{\omega}$ on $\tilde{M}$ is called **$\mathbb{Z}$-automorphic** if the deck transform group mapping M_i to M_{i+1} acts on $(\tilde{M}, \tilde{\omega})$ by homotheties. To obtain such a form **we need to find a Kähler form $\hat{\omega}$ on $\hat{B}$ such that $\hat{\omega}\big|_{B_0}$ is equal to $\Psi^* \hat{\omega}$ in a neighbourhood of the boundary of B_0.** If this is true, Ψ acts by homotheties in a neighbourhood of S. Then the restriction of $\hat{\omega}$ to $\hat{B} \backslash B_0 = M_i$ can be extended to a $\mathbb{Z}$-automorphic Kähler form on $\tilde{M} = \bigcup_{i \in \mathbb{Z}} M_i$.

Step 2: Start with a Kähler form $\hat{\omega}$ on $\hat{B}$ (it always exists, because $\hat{B}$ is a blow-up of a ball). Using the local dd^c-lemma, we can find a smooth function φ on B_0 such that $dd^c \varphi = \hat{\omega}\big|_{B_0}$. Solving the appropriate elliptic equation with boundary conditions, **we can assume that $B_0 = \varphi^{-1}(-\infty, 0)$.**

Brunella's theorem (2)

Step 3: Let $\pi_*\widehat{\omega}$ be the pushforward of $\widehat{\omega}$, considered to be a current on B. Clearly, $\pi_*\widehat{\omega}$ **is closed and positive.** Using the dd^c-lemma for currents, **we obtain $\pi_*\widehat{\omega} = dd^c f$,** where f is a plurisubharmonic function on B that is smooth outside of the singularities of π. We also assume that $f = const$ on ∂B. Let $f_1 := (\Psi^{-1})^*\varphi$, where φ is the Kähler potential on B_0 constructed in Step 2, and $\Psi: B \longrightarrow B_0$ the biholomorphic map used to construct M.

Step 4: Let S be the boundary of B. Rescaling f if necessary and adding a constant, we may assume that $-\varepsilon < f|_S < 0$ and $|df|\big|_S \ll \varepsilon$. Let A be a sufficiently big positive number, and $0 < \delta \ll \varepsilon$. Then the regularized maximum $\max_\delta(f, Af_1)$ is equal to Af_1 in a very small neighbourhood of S (because f is negative on S and $f_1 = 0$ on S), and equal to f in a small neighbourhood V of $S_1 := Af_1^{-1}(-2\varepsilon)$ because $|df| \ll A|df_1|$ and as Af_1 goes to -2ε, f does not go much below $-\varepsilon$.

Step 5: Replacing $\widehat{\omega}$ by $dd^c\max_\delta(f, f_1)$ on the annulus between S and S_1, we obtain a Kähler form $\widehat{\omega}_1$. Since $\max_\delta(f, f_1) = f_1$ in a neighbourhood of S, the map $\Psi: (B, \widehat{\omega}_1) \longrightarrow (B_0, \widehat{\omega}_1)$ acts on a neighbourhood of S mapping the metric $\widehat{\omega}_1$ isometrically to a neighbourhood of $\Psi(S)$ with the metric $A\widehat{\omega}_1$. **As indicated in Step 1, this construction gives an LCK metric on M.** ■