Topologia das Variedades: class exercise 2

Exercise 2.1. Let M be a smooth manifold, and $\operatorname{Tot} TM$ the total space of its tangent bundle. Prove that the manifold $\operatorname{Tot} TM$ is orientable, or find a counterexample.

Exercise 2.2. Let M be an orientable compact n-manifold. Prove that $H^n(M) = \mathbb{R}$.

Exercise 2.3. Let G be a finite group acting freely on a compact manifold M, and let $\Lambda^*(M)^G$ denote the G-invariant differential forms. Prove that the cohomology of the complex

$$\Lambda^0(M)^G \longrightarrow \Lambda^1(M)^G \longrightarrow \Lambda^2(M)^G \longrightarrow ...$$

are equal to the de Rham cohomology of the quotient M/G.

Exercise 2.4. Let T^n be n-dimensional compact torus. Consider the action if T^n on itself, $x(y) = x + y$. Denote by $\Lambda^k(T^n)^{T^n}$ the space of T^n-invariant k-forms on T^n. Prove that $\Lambda^k(T^n)^{T^n} = \Lambda^k(\mathbb{R}^n) = H^k(T^n)$.

Exercise 2.5. Let k be an odd number. Recall that **an odd derivation** of the de Rham algebra $\Lambda^*(M)$ is a map $\delta : \Lambda^*(M) \longrightarrow \Lambda^{*+k}(M)$ which satisfies $\delta(\alpha \wedge \beta) = \delta(\alpha) \wedge \beta + (-1)^{\tilde{\alpha}} \alpha \wedge \delta(\beta)$. Prove that an anticommutator $\delta_1\delta_2 + \delta_2\delta_1$ of two odd derivations δ_1, δ_2 of $\Lambda^*(M)$ is an even derivation, that is, satisfies the usual Leibitz identity.