Topologia das Variedades: class exercise 3

Exercise 3.1. Consider the product of two spheres $M := S^n \times S^m$ equipped with the action of the group $G := SO(n+1) \times SO(m+1)$. Find the space of G-invariant differential forms on M.

Exercise 3.2. Consider the complex projective space $M = \mathbb{C}P^2$, equipped with an action of the group $G = U(2)$.

a. Find the space of G-invariant differential forms on M.

b. Prove that the de Rham cohomology of $H^*(\mathbb{C}P^2)$ are isomorphic to the algebra of G-invariant forms. Find this space and describe its multiplicative structure.

c. Prove that any diffeomorphism of $\mathbb{C}P^2$ preserves the orientation.

Hint. Use the algebra structure on the de Rham cohomology.

Definition 3.1. Symmetric space is a complete Riemannian manifold M such that for each $x \in M$ there exists an isometry τ_x fixing x and acting on its tangent space T_xM as $-\mathsf{Id}$.

Exercise 3.3. Prove that the following spaces admit a Riemannian structure making them symmetric.

a. Sphere S^n and a real projective space $\mathbb{R}P^n$ (for all n).

b. Complex projective space $\mathbb{C}P^n$.

c. Grassmann space of all k-dimensional planes in $\mathbb{R}^n$.

Exercise 3.4. Prove that the space $SO(1,n)/SO(n)$ admits an $SO(n,1)$-invariant metric, which is unique up to a constant multiplier. Prove that it is symmetric.

Remark 3.1. The space $SO(1,n)/SO(n)$ with this metric is called **the hyperbolic space**.

Exercise 3.5. Let M be a connected symmetric space. Prove that its isometry group acts on M transitively.

Exercise 3.6. Let M be a symmetric space, $M = G/H$, where G is its isometry group and H the stabilizer group of a point $x \in M$.

a. Prove that all G-invariant differential forms on M are even.

b. Use this to show that the de Rham cohomology of a symmetric space G/H, with G compact, are isomorphic to the space of G-invariant differential forms.