Topologia das Variedades

Cohomology, lecture 1: Grassmann algebra

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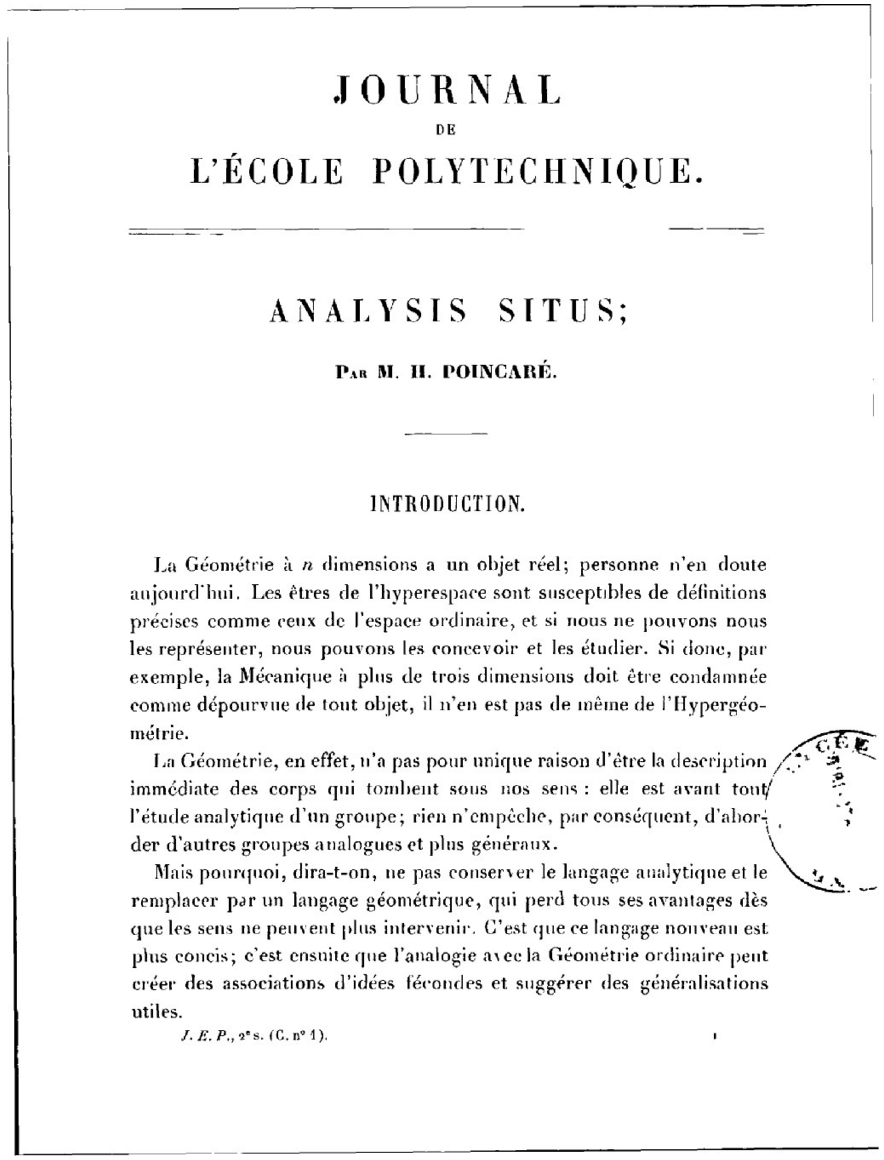
Analysis situs



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DE

L'ÉCOLE POLYTECHNIQUE.

ANALYSIS SITUS;

PAR M. H. POINCARÉ.

INTRODUCTION.

La Géométrie à n dimensions a un objet réel; personne n'en doute aujourd'hui. Les êtres de l'hyperespace sont susceptibles de définitions précises comme ceux de l'espace ordinaire, et si nous ne pouvons nous les représenter, nous pouvons les concevoir et les étudier. Si donc, par exemple, la Mécanique à plus de trois dimensions doit être condamnée comme dépourvue de tout objet, il n'en est pas de même de l'Hypergéométrie.

La Géométrie, en effet, n'a pas pour unique raison d'être la description immédiate des corps qui tombent sous nos sens : elle est avant tout l'étude analytique d'un groupe; rien n'empêche, par conséquent, d'aborder d'autres groupes analogues et plus généraux.

Mais pourquoi, dira-t-on, ne pas conserver le langage analytique et le remplacer par un langage géométrique, qui perd tous ses avantages dès que les sens ne peuvent plus intervenir. C'est que ce langage nouveau est plus concis; c'est ensuite que l'analogie avec la Géométrie ordinaire peut créer des associations d'idées fécondes et suggérer des généralisations utiles.

J. E. P., 2e s. (C. n° 1). 1

"Analysis Situs" is a seminal mathematics paper that Henri Poincaré published in 1895. Poincaré published five supplements to the paper between 1899 and 1904.

Bilinear maps

DEFINITION: Let U, V, W be vector spaces over k. A map $U \times V \stackrel{\mu}{\to} W$, $u, v \mapsto \mu(u, v)$ is called **bilinear** if for all $u \in U$ and $v \in V$ the maps $\mu(u, \cdot) : V \to W \quad \mu(\cdot, v) : U \to W$ are linear.

REMARK: Clearly, a linear combination of bilinear maps is bilinear. Then **the space** $\operatorname{Bil}(U \times V, W)$ **of bilinear maps** $U \times V \to W$ **is a vector space**.

DEFINITION: **Bilinear form** on V is a bilinear map $V \times V \stackrel{\mu}{\to} k$. **Bilinear symmetric form** is a form which satisfies $\mu(x, y) = \mu(y, x)$ for all $x, y \in V$. **Bilinear anti-symmetric formm** or **bilinear skew-symmetric form** is a form which satisfies $\mu(x, y) = -\mu(y, x)$. We denote the first by $\operatorname{Sym}^2 V^*$, and the second by $\Lambda^2 V^*$.

Tensor product

DEFINITION: Let S be a set. Define **vector space, freely generated by** S as the space of functions $\psi: S \longrightarrow k$ which are equal zero outside of a finite subset $\mathrm{Sup}_\psi \subset S$.

DEFINITION: Let V, V' be vector spaces over k, and W a vector space freely generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subspace generated by combinations $av \otimes v' - v \otimes av'$, $a(v \otimes v') - (av) \otimes v'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$, where $a \in k$. Define **the tensor product** $V \otimes_k V'$ as a quotient vector space W/W_1.

PROPOSITION: For any vector spaces V, V', R, there is a natural identification $\mathrm{Hom}(V \otimes_k V', R) = \mathrm{Bil}(V \times V', R)$.

Proof: Clearly, any bilinear map $\rho \in \mathrm{Bil}(V \times V', R)$ defines a linear map $\tilde{\rho}: W \longrightarrow R$, and $\tilde{\rho}$ vanishes on W_1. This gives a map $\mathrm{Bil}(V \times V', R) \longrightarrow \mathrm{Hom}(V \otimes_k V', R)$. Inverse map takes $\tau \in \mathrm{Hom}(V \otimes_k V', R)$ and interprets it as a bilinear map in $\mathrm{Bil}(V \times V', R)$. ■

COROLLARY: For finite-dimensional V, V', one has $V \otimes_k V' = \mathrm{Bil}(V \times V', k)^*$.

Dimension of the tensor product

CLAIM: Dimension of $\operatorname{Bil}(V \times V', k)$ is equal to $\dim V \dim V'$.

Proof. Step 1: Let $\{\lambda_i\}$ be a basis in V^* and $\{\lambda'_i\}$ a basis in V'. Denote by $\{v_i\}$ $\{v'_i\}$ the dual basis in V, V'. Then $\lambda_i \lambda'_j$ can be interpreted as vectors in $\operatorname{Bil}(V \times V', k)$. These vectors are clearly linearly independent: indeed

$$\sum_{i,j} a_{ij} \lambda_i \lambda'_j (v_p, v'_q) = a_{pq}.$$

This gives $\dim \operatorname{Bil}(V \times V', k) \geqslant \dim V \dim V'$.

Step 2: On the other hand, $\dim V \otimes V' \leqslant \dim V \dim V'$, because it is generated by $v_p \otimes v_q$, hence $\dim \operatorname{Bil}(V \times V', k) \leqslant \dim V \dim V'$. ■

COROLLARY: Let $\{x_i\}$ and $\{y_i\}$ be bases in V, W. **Then $\{x_i \otimes y_j\}$ is a basis in $V \otimes_k W$.** ■

Algebra over a field

Fix a ground field k. Recall that a map $(V_1 \times V_2) \xrightarrow{\mu} V_3$ of vector spaces is called **bilinear** of for any $v_1 \in V_1$, $v_2 \in V_2$, the maps $\mu(v_1, \cdot): V_2 \longrightarrow V_3$, $\mu(\cdot, v_2): V_1 \longrightarrow V_3$ (one element is fixed) is k-linear.

To express this, we use the tensor product sign, and write $\mu: V_1 \otimes V_2 \longrightarrow V_3$.

DEFINITION: Let A be a vector space over k, and $\mu: A \otimes A \longrightarrow A$ a bilinear map (called **"multiplication"**). The pair (A, μ) is called **algebra over a field k** if μ is **associative**: $\mu(a_1, \mu(a_2, a_3)) = \mu(\mu(a_1, a_2), a_3))$. The product in algebra is written as $a \cdot b$ or ab. If, in addition, there is an element $1 \in A$ such that $\mu(1, a) = \mu(a, 1) = a$ for all $a \in A$, this element is called **unity**, a and A **an algebra with unity**.

DEFINITION: A **homomorphism** of algebras $r: A \longrightarrow A'$ is a linear map which is compatible with a product. **Isomorphism** of algebras is an invertible homomorphism. **Subalgebra** of an algebra A is a vector subspace which is closed under multiplication.

Tensor algebra

Let $V \times W$ map to $V \otimes W$ by putting v, w to $v \otimes w$. Clearly, this map is bilinear. Similarly, one has a bilinear map $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes m+n}$ putting $x_1 \otimes ... \otimes x_n, y_1 \otimes ... \otimes y_n$ to $x_1 \otimes ... \otimes x_n \otimes y_1 \otimes ... \otimes y_n$.

DEFINITION: Let V be a vector space over k. **Tensor algebra**, or **free algebra generated by V** is $T(V) := \bigoplus_i V^{\otimes^i}$ equipped with the multiplicative structure defined above,

EXERCISE: Prove that $T(V^*)$ **is the algebra of polylinear forms on V defined above.**

REMARK: If $x_1, ..., x_r$ is a basis in V, then **the basis in $V^{\otimes^n}$ is formed by all different monomials** of the form $x_{i_1} \otimes x_{i_2} \otimes ... \otimes x_{i_n}$.

Two-sided ideals

DEFINITION: Let A be an algebra and $J \subset A$ its subspace. Then J is caled **left ideal** if for all $a \in A, j \in J$, one has $ja \in J$, and **right ideal** if one has $aj \in J$. J is called **two-sided ideal** if is is both right and left ideal.

REMARK: Let $J \subset A$ be a two-sided ideal, $x, y \in A/J$ some vectors, and $\tilde{x}, \tilde{y}$ Define the product $xy \in A/J$ by putting $x \cdot y$ to the class represented by $\tilde{x}, \tilde{y}$. Since $ja \in J$ and $aj \in J$, **this gives a bilinear map** $A/J \otimes A/J \longrightarrow A/J$, defining an associative multiplicative structure on A/J.

CLAIM: In these assumptions, A/J **is an algebra,** with the product defined as above.

EXERCISE: Prove it.

Algebra defined by generators and relations

DEFINITION: Let V be a vector space over k **("the space of generators")**, and $W \subset T(V)$ another vector space **("the space of relations")**. Consider a quotient A of $T(V)$ by the subspace $T(V)WT(V)$ generated by the vectors $v \otimes w \otimes v'$, where $w \in W$ and $v, v' \in T(V)$.

CLAIM: There is a natural product structure on the space $A := \frac{T(V)}{T(V)WT(V)}$.

Proof: $T(V)WT(V)$ is a 2-sided ideal. ■

DEFINITION: In this situation, we say that A **is an algebra defined by generators and relations**.

EXERCISE: Prove that any algebra can be defined by generators and relations.

DEFINITION: An algebra is called **finitely generated** if it can be defined by generators and relations, and the space of generators is finitely-dimensional. An algebra is called **finitely presented** if the space W of relations is finitely-dimensional.

Graded algebras

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: A subspace $W \subset A^*$ of a graded algebra is called **graded** if W is a direct sum of components $W^i \subset A^i$.

EXERCISE: Let $W \subset T(V)$ be a graded subspace. Prove that then **the algebra generated by V with relation space W is also graded.**

The Grassmann algebra

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Antisymmetric tensors

DEFINITION: Let $V^{\otimes^n}$ be n-th product of V with itself, equipped with the natural symmetric group Σ_n-action exchanging the tensor components. A tensor $\psi \in V^{\otimes^n}$ is called **antisymmetric** if for any permutation $\sigma \in \Sigma_n$ we have $\sigma(\psi) = (-1)^{\tilde{\sigma}}\psi$, and **symmetric** if $\sigma(\psi) = \psi$. We denote the space of all antisymmetric tensors by $\tilde{\Lambda}^n V$ and the space of symmetric tensors by $\widetilde{\operatorname{Sym}}^n V$.

Theorem 1: Let V be a vector space, $\Lambda^n V$ the n-th component of its Grassmann algebra, and $\operatorname{Sym}^n V$ the n-th component of its symmetric algebra. Then **$\Lambda^n V$ is naturally identified with $\tilde{\Lambda}^n V$, and $\operatorname{Sym}^n V$ with $\widetilde{\operatorname{Sym}}^n V$:** the projection from $V^{\otimes n}$ to $\Lambda^n V$ (or $\operatorname{Sym}^n V$) induces an isomorphism from $\tilde{\Lambda}^n V$ (or $\widetilde{\operatorname{Sym}}^n V$) to $\Lambda^n V$ (or $\operatorname{Sym}^n V$).

Antisymmetrization

The natural map from $\Lambda^n V$ to $\tilde{\Lambda}^n V$ is given by **antisymmetrization**

$$\operatorname{Alt}(x_1 \otimes ... \otimes x_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}.$$

Its properties:

1. Clearly, $\operatorname{im}\operatorname{Alt}$ lies in the space of antisymmetric tensors, and $\operatorname{Alt}(\eta - (-1)^{\tilde{\sigma}}\eta) = 0$, hence Alt **defines a map from Grassmann algebra to the space of antisymmetric tensors.**

2. The natural **projection from antisymmetric tensors to the Grassmann algebra is inverse to** Alt**.**

Grassmann algebra: dimension of components

REMARK: For linearly independent vectors $x_1, ..., x_k$, the antisymmetrization $x_1 \wedge x_2 \wedge ... \wedge x_k := \frac{1}{k!}\sum(-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ is non-trivial. Indeed, the monomials $x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ are linearly independent in $V^{\otimes k}$. **This implies** $\dim \Lambda^k V \geqslant \binom{\dim V}{k}$.

CLAIM: $\dim \Lambda^k V = \binom{\dim V}{k}$**, and** $\dim \Lambda^* V = 2^{\dim V}$**.**

Proof: Let $x_1, ..., x_n$ be a basis in V. Then **the space $\Lambda^k V$ is generated by antisymmetric tensors $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k}$, $i_1 < i_2 < ... < i_k$, which are all linearly independent.** ■

Tensor product of vector bundles

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v\otimes v'$, with $v\in V, v'\in V'$, and $W_1\subset W$ a subgroup generated by combinations $rv\otimes v' - v\otimes rv'$, $(v_1+v_2)\otimes v' - v_1\otimes v' - v_2\otimes v'$ and $v\otimes(v'_1+v'_2) - v\otimes v'_1 - v\otimes v'_2$. Define **the tensor product** $V\otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r\cdot v\otimes v'\mapsto (rv)\otimes v'$ **defines an R-module structure on $V\otimes_R V'$.**

DEFINITION: **Tensor product** of vector bundles is a tensor product of the corresponding sheaves of modules.

EXERCISE: Let B and B' ve vector bundles on M, $B|_x$, $B'|_x$ their fibers, and $B\otimes_{C^\infty M} B'$ their tensor product. **Prove that $B\otimes_{C^\infty M} B'|_x = B|_x\otimes_{\mathbb R} B'|_x$.**

Cotangent bundle

DEFINITION: Let M be a smooth manifold, TM the tangent bundle, and $\Lambda^1 M = T^*M$ its dual bundle. It is called **cotangent bundle** of M. Sections of T^*M are called **1-forms** or **covectors** on M. For any $f \in C^\infty M$, consider a functional $TM \longrightarrow C^\infty M$ obtained by mapping $X \in TM$ to a derivation of f: $X \longrightarrow D_X(f)$. Since this map is linear in X, it defines a section $df \in T^*M$ called **the differential** of f.

CLAIM: $\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$.

Proof: Locally in coordinates $x_1, ..., x_n$ this is clear, because the covectors $dx_1, ..., dx_n$ give a basis in T^*M dual to the basis $\frac{d}{dx_1}, ..., \frac{d}{dx_n}$ in TM. To obtain the same statement globally, use the partition of unity. ■

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.