Topologia das Variedades

Cohomology, lecture 2: de Rham differential

Misha Verbitsky

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The Grassmann algebra (reminder)

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

Antisymmetrization (reminder)

The natural embedding from $\Lambda^n V$ to $T^{\otimes n} V$ is given by **antisymmetrization**

$$\mathrm{Alt}(x_1 \wedge ... \wedge x_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (-1)^{\tilde{\sigma}} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(n)}.$$

Its properties:

1. Clearly, $\operatorname{im} \mathrm{Alt}$ lies in the space of antisymmetric tensors, and $\mathrm{Alt}(\eta - (-1)^{\tilde{\sigma}} \eta) = 0$, hence Alt **defines a map from Grassmann algebra to the space of antisymmetric tensors.**

2. The natural **projection from antisymmetric tensors to the Grassmann algebra is inverse to** Alt**.**

3. This map **defines an isomorphism of the Grassmann algebra and the space of antisymmetric tensors.**

Grassmann algebra: dimension of components (reminder)

REMARK: For linearly independent vectors $x_1, ..., x_k$, the antisymmetrization $x_1 \wedge x_2 \wedge ... \wedge x_k := \frac{1}{k!}\sum (-1)^{\tilde\sigma} x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ is non-trivial. Indeed, the monomials $x_{\sigma(1)} \otimes ... \otimes x_{\sigma(k)}$ are linearly independent in $V^{\otimes k}$. **This implies** $\dim \Lambda^k V \geqslant \binom{\dim V}{k}$.

CLAIM: $\dim \Lambda^k V = \binom{\dim V}{k}$, **and** $\dim \Lambda^* V = 2^{\dim V}$.

Proof: Let $x_1, ..., x_n$ be a basis in V. Then **the space $\Lambda^k V$ is generated by antisymmetric tensors** $x_{i_1} \wedge x_{i_2} \wedge ... \wedge x_{i_k}$, $i_1 < i_2 < ... < i_k$, **which are all linearly independent.** ∎

Tensor product of vector bundles

DEFINITION: Let V, V' be R-modules, W a free abelian group generated by $v \otimes v'$, with $v \in V, v' \in V'$, and $W_1 \subset W$ a subgroup generated by combinations $rv \otimes v' - v \otimes rv'$, $(v_1 + v_2) \otimes v' - v_1 \otimes v' - v_2 \otimes v'$ and $v \otimes (v'_1 + v'_2) - v \otimes v'_1 - v \otimes v'_2$. Define **the tensor product** $V \otimes_R V'$ as a quotient group W/W_1.

EXERCISE: Show that $r \cdot v \otimes v' \mapsto (rv) \otimes v'$ **defines an R-module structure on $V \otimes_R V'$.**

DEFINITION: Tensor product of vector bundles is a tensor product of the corresponding sheaves of modules.

EXERCISE: Let B and B' ve vector bundles on M, $B|_x$, $B'|_x$ their fibers, and $B \otimes_{C^\infty M} B'$ their tensor product. **Prove that $B \otimes_{C^\infty M} B'|_x = B|_x \otimes_{\mathbb{R}} B'|_x$.**

Cotangent bundle

DEFINITION: Let M be a smooth manifold, TM the tangent bundle, and $\Lambda^1 M = T^*M$ its dual bundle. It is called **cotangent bundle** of M. Sections of T^*M are called **1-forms** or **covectors** on M. For any $f \in C^\infty M$, consider a functional $TM \longrightarrow C^\infty M$ obtained by mapping $X \in TM$ to a derivation of f: $X \longrightarrow D_X(f)$. Since this map is linear in X, it defines a section $df \in T^*M$ called **the differential** of f.

CLAIM: $\Lambda^1 M$ is generated as a $C^\infty M$-module by $d(C^\infty M)$.

Proof: Locally in coordinates $x_1, ..., x_n$ this is clear, because the covectors $dx_1, ..., dx_n$ give a basis in T^*M dual to the basis $\frac{d}{dx_1}, ..., \frac{d}{dx_n}$ in TM. To obtain the same statement globally, use the partition of unity. ■

DEFINITION: Let M be a smooth manifold. **A bundle of differential i-forms on M** is the bundle $\Lambda^i T^*M$ of antisymmetric i-forms on TM. It is denoted $\Lambda^i M$.

REMARK: $\Lambda^0 M = C^\infty M$.

De Rham algebra

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T^*_x M$.** This identification is compatible with the Grassmann product.

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several dt_i: $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$ $i_1 < i_2 < ... < i_k$. Then α is called **a coordinate monomial**.

De Rham differential

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \dots$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Derivations determined by their values on generators

REMARK: A map d on a graded algebra which satisfies the Leibnitz rule $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde{\eta}} \eta \wedge d(\xi)$ is called **an odd derivation**.

REMARK: The following two lemmas are needed to prove uniqueness of de Rham differential.

LEMMA: Let $A = \bigoplus A^i$ be a graded algebra, $B \subset A$ a set of multiplicative generators, and $D_1, D_2 : A \longrightarrow A$ two odd derivations which are equal on B. **Then $D_1 = D_2$.** ■

LEMMA: $\Lambda^* M$ is generated by $C^\infty M$ and $d(C^\infty M)$.

Proof: By definition, $\Lambda^* M$ is generated by $\Lambda^0 M = C^\infty M$ and $\Lambda^1 M$. However, $d(C^\infty M)$ generate $\Lambda^1 M$, as shown above. ■

De Rham differential: uniqueness and existence

THEOREM:
De Rham differential is uniquely determined by these axioms.

Proof: De Rham differential is an odd derivation. Its value on $C^\infty M$ is defined by the first axiom. On $d(C^\infty M)$ de Rham differential valishes, because $d^2 = 0$.
■

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, α_i coordinate monomials, and $\alpha := \sum f_i \alpha_i$. Define $d(\alpha) := \sum_i \sum_j \frac{df_i}{dt_j} dt_j \wedge \alpha_i$.

EXERCISE:
Check that d satisfies the properties of de Rham differential.

COROLLARY: De Rham differential exists on any smooth manifold.

Proof: Locally, de Rham differential d exists, as follows from the construction above. Cover M by open sets where d is defined. Since d is unique, it is compatible on intersections of these open sets. **Therefore, it is well defined globally on M.** ■

Georges de Rham (1903-1990)

Georges de Rham (with Marcel Berger)

Integration

DEFINITION: Let M be an n-dimensional manifold. Clearly, $\Lambda^n(M)$ is a rank 1 vector bundle. This bundle is trivial if and only if it has a non-vanishing section η. Let's call two such sections η, η' **equivalent** if $\eta = f\eta'$, where $f \in C^\infty M$ is positive everywhere. A choice of equivalence class of such sections is called **an orientation** on M, and M is called **oriented**. A section of $\Lambda^n(M)$ which belongs to this class is called **a differential n-form, positive everywhere**, or **a volume form**.

THEOREM: Let M be an n-dimensional compact oriented manifold. Denote by $\Lambda^n(M)$ the space of sections of the vector bundle $\Lambda^n(M)$. **Then there exists a linear functional $\int_M : \Lambda^n(M) \longrightarrow \mathbb{R}$** which is positive on volume forms and invariant under the $\operatorname{Diff}(M)$ (diffeomorphism group of M) action. Moreover, **such a functional is unique up to a positive constant multiplier.**

REMARK: To fix the normalization, it would suffice to take a diffeomorphism from a unit cube to $U \subset M$ which can be extended to a small neighbourhood of the cube. Consider the pushforward η of the standard volume form $dx_1 \wedge ... \wedge dx_n$ from the cube to M, and let $\int_M \eta = 1$. It is not hard to see that **all such maps from the cube to $U \subset M$ are diffeomorphism-conjugate**, hence **this choice defines the functional $\int_M$ uniquely.**

Stokes' theorem and Poincaré lemma

DEFINITION: The map $\int_M : \Lambda^n(M) \longrightarrow \mathbb{R}$ defined above is called **the integral** of the differential form.

Stokes' theorem: (Poincarè)
Let η be $n-1$-form on n-manifold M with a boundary ∂M. **Then**

$$\int_M d\eta = \int_{\partial M} \eta.$$

Existence and uniqueness of the integral and Stokes' theorems are left as exercises for now.

Poincaré lemma:
De Rham cohomology $H^i(B)$ of an open ball B vanish for $i > 0$.