Topologia das Variedades

Cohomology, lecture 3: the Lie derivative

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The Grassmann algebra (reminder)

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde x$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde x \tilde y} y \wedge x$.

De Rham algebra (reminder)

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T_x^* M$.** This identification is compatible with the Grassmann product.

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several dt_i: $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$ $i_1 < i_2 < ... < i_k$. Then α is called **a coordinate monomial**.

De Rham differential (reminder)

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} \dots$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. **(Graded Leibniz identity)** $d(\eta\wedge\xi) = d(\eta)\wedge\xi + (-1)^{\tilde\eta}\eta\wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i}M$ is **an even form,** and $\eta\in\lambda^{2i+1}M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Graded algebras (reminder)

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

EXAMPLE: The tensor algebra $T(V)$ and the polynomial algebra $\operatorname{Sym}^*(V)$ are obviously graded.

DEFINITION: A subspace $W \subset A^*$ of a graded algebra is called **graded** if W is a direct sum of components $W^i \subset A^i$.

EXERCISE: Let $W \subset T(V)$ be a graded subspace. Prove that then **the algebra generated by V with relation space W is also graded.**

Superalgebras

DEFINITION: Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra over a field. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{ij} ba$ for all $a \in A^i, b \in A^j$.

EXAMPLE: Grassmann algebra $\Lambda^* V$ is clearly supercommutative.

DEFINITION: Let A^* be a graded commutative algebra, and $D: \ A^* \longrightarrow A^{*+i}$ be a map which shifts grading by i. It is called a **graded derivation**, if $D(ab) = D(a)b + (-1)^{ij} aD(b)$, for each $a \in A^j$.

REMARK: If i is even, graded derivation is a usual derivation. If it is even, it an odd derivation.

DEFINITION: Let M be a smooth manifold, and $X \in TM$ a vector field. Consider an operation of **convolution with a vector field** $i_X: \ \Lambda^i M \longrightarrow \Lambda^{i-1} M$, mapping an i-form α to an $(i-1)$-form $v_1, ..., v_{i-1} \longrightarrow \alpha(X, v_1, ..., v_{i-1})$

EXERCISE: Prove that i_X is an odd derivation.

Supercommutator

DEFINITION: Let A^* be a graded vector space, and $E: A^* \longrightarrow A^{*+i}$, $F: A^* \longrightarrow A^{*+j}$ operators shifting the grading by i, j. Define **the supercommutator** $\{E,F\} := EF - (-1)^{ij} FE$.

DEFINITION: An endomorphism of a graded vector space which shifts grading by i is called **even** if i is even, and **odd** otherwise.

EXERCISE: Prove that the supercommutator satisfies **graded Jacobi identity,**

$$\{E,\{F,G\}\} = \{\{E,F\},G\} + (-1)^{\tilde{E}\tilde{F}}\{F,\{E,G\}\}$$

where $\tilde{E}$ and $\tilde{F}$ are 0 if E, F are even, and 1 otherwise.

REMARK: There is a simple mnemonic rule which allows one to remember a superidentity, if you know the commutative analogue. **Each time when in commutative case two letters E, F are exchanged, in supercommutative case one needs to multiply by $(-1)^{\tilde{E}\tilde{F}}$.**

EXERCISE: **Prove that a supercommutator of superderivations is again a superderivation.**

Pullback of a differential form

DEFINITION: Let $M \xrightarrow{\varphi} N$ be a morphism of smooth manifolds, and $\alpha \in \Lambda^i N$ be a differential form. Consider an i-form $\varphi^*\alpha$ taking value

$$\alpha\Big|_{\varphi(m)}(D_\varphi(x_1), ...D_\varphi(x_i))$$

on $x_1, ..., x_i \in T_m M$. It is called **the pullback of α**. If $M \xrightarrow{\varphi} N$ is a closed embedding, the form $\varphi^*\alpha$ is called **the restriction** of α to $M \hookrightarrow N$.

LEMMA: (*) Let $\Psi_1, \Psi_2 : \Lambda^* N \longrightarrow \Lambda^* M$ be two maps which satisfy graded Leibnitz identity, supercommutes with de Rham differential, and satisfy $\Psi_1|_{C^\infty M} = \Psi_2|_{C^\infty M}$. **Then $\Psi_1 = \Psi_2$.**

Proof: The algebra $\Lambda^* M$ is generated multiplicatively by $C^\infty M$ and $d(C^\infty M)$; restrictions of Ψ_i to these two spaces are equal. ■

CLAIM: Pullback commutes with the de Rham differential.

Proof: Let $d_1, d_2 : \Lambda^* N \longrightarrow \Lambda^{*+1} M$ be the maps $d_1 = \varphi^* \circ d$ and $d_2 = d \circ \varphi^*$. **These maps satisfy the Leibnitz identity, and they are equal on $C^\infty M$.** The super-commutator $\delta := \{d_i, d\}$ is equal to $d \circ \varphi^* \circ d$, it commutes with d, and equal 0 on functions. By Lemma (*), $\delta = 0$. Then d_i supercommutes with d. Applying Lemma (*) again, we obtain that $d_1 = d_2$. ■

Lie derivative

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along v** if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra.

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v|_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

EXERCISE: Prove that $\{d, \{d, E\}\} = 0$ **for each** $E \in \operatorname{End}(\Lambda^* M)$**.**

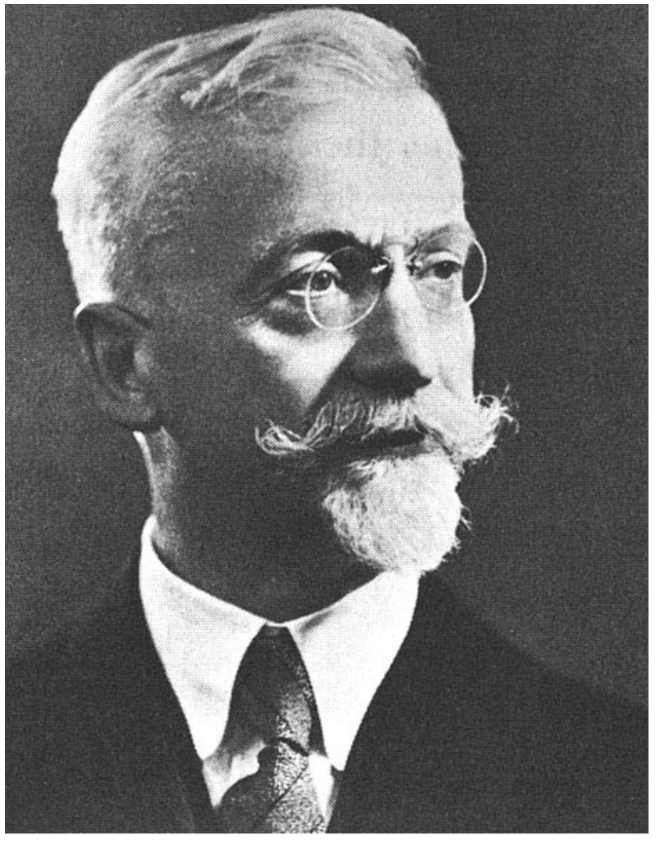
THEOREM: (Cartan's formula) Let i_v be a convolution with a vector field, $i_v(\eta) = \eta(v, \cdot, \cdot, ..., \cdot)$ **Then $\{d, i_v\}$ is equal to the Lie derivative along v.**

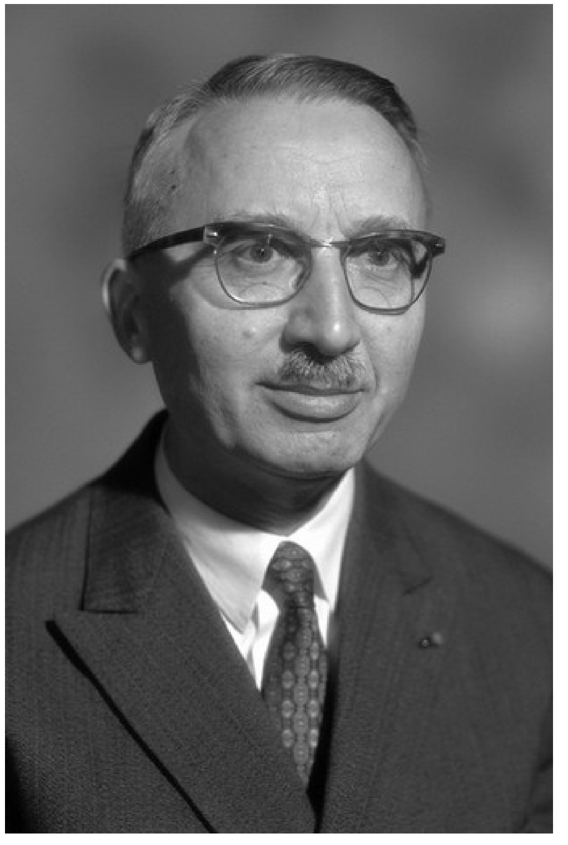
Proof: $\{d, \{d, i_v\}\} = 0$ by the lemma above. A supercommutator of two graded derivations is a graded derivation. Finally, $\{d, i_v\}$ acts on functions as $i_v(df) = \langle v, df \rangle$. ■

Cartan's magic formula: $d \circ i_x + i_x \circ d = \mathrm{Lie}_x$.

Which Cartan?

Élie Cartan (1869-1951) Henri Cartan (1904-2008)





Élie Cartan?
(Robert Bryant and Dick Palais Mathoverflow)

Henri Cartan?
(S.S. Chern: Lectures on differential geometry)

Flow of diffeomorphisms

DEFINITION: Let $f: M\times [a,b] \longrightarrow M$ be a smooth map such that for all $t\in [a,b]$ the restriction $f_t := f\Big|_{M\times\{t\}} : M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f\in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c} : C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^*\frac{dV_t}{dt}|_{t=c}f$. **Then $f \longrightarrow (V_t^{-1})^* \frac{d}{dt}V_t^* f$ is a derivation** (that is, a vector field).

Proof: $\frac{d}{dt}V_t^*(fg) = V_t^*(f)\frac{d}{dt}V_t^*g + \frac{d}{dt}V_t^*fV_t^*(g)$ by the Leignitz rule, giving

$$(V_t^{-1})^*\frac{d}{dt}V_t^*(fg) = f(V_t^{-1})^*\frac{d}{dt}V_t^*g + g(V_t^{-1})^*\frac{d}{dt}V_t^*f.$$

■

DEFINITION: The vector field $\frac{d}{dt}V_t|_{t=c}$ is called **the vector field tangent to a flow of diffeomorphisms V_t at $t=c$.**

CLAIM: Let V_t be a flow of diffeomorphisms and X_t the corresponding vector field. **Then for any $\eta\in\Lambda^*M$, one has $\frac{d}{dt}V_t^*(\eta) = \operatorname{Lie}_{X_t}(\eta)$.**

Proof: The operators $\frac{d}{dt}V_t^*$ and Lie_{X_t} are equal on functions, satisfy the Leibitz identity and commute with d. ■

Flow of diffeomorphisms obtained from vector fields

EXERCISE: Let M be a compact manifold, and $\Psi: C^\infty M \longrightarrow C^\infty M$ is a ring automorphism. Prove that Ψ **is induced by an action of a diffeomorphism of M.**

THEOREM: Let M be a compact manifold, and $X_t \in TM$ a family of vector fields smoothly depending on $t \in [0, a]$. **Then there exists a unique diffeomorphism flow V_t, $t \in [0, a]$, such that $V_0 = \mathrm{Id}$ and $\frac{d}{dt} V_t^* = X_t$.**

Proof. Step 1: Given $f \in C^\infty M$, we can solve an equation $\frac{d}{dt} W_t(f) = \mathrm{Lie}_{X_t}(f)$ (here $\mathrm{Lie}_{X_t}(f)$ denotes the derivative along the vector field). The solution $W_t(f)$ exists for all $t \in [0, x]$ and is unique by Peano theorem on existence and uniqueness of solutions of ODE,

Step 2: Since

$$\frac{d}{dt} W_t(fg) = \mathrm{Lie}_{X_t}(f)g + \mathrm{Lie}_{X_t}(g)f = \frac{d}{dt}(W_t(f)W_t(g)),$$

W_t is multiplicative. Also, it is invertible. Applying the previous exercise, we obtain that W_t is a diffeomorphism. ■

For another proof see Chapter 5 of *Arnold V.I., Ordinary Differential Equations*

Flow of diffeomorphisms from vector fields on maifolds with boundary

REMARK: The same result **holds for compact manifolds with boundary if the vector field is tangent to the boundary.** The proof is the same.

REMARK: The same result holds if M is a complete Riemannian manifold and X_t is C-Lipschitz for all t, that is, satisfies $\Big||X_t|_x - |X_t|_y\Big| \leqslant Cd(x,y)$. It also follows from the Peano theorem.

Let M be a compact manifold with boundary ∂M, and $X_t \in TM$ a family of vector fields smoothly depending on $t \in [0,a]$. Suppose that $X_t|_{\partial M}$ is pointed inside, and not outside. We can embed M to $M_1 = M \cup \partial M \times [0,1]$ obtained by gluing $\partial M \subset M$ to $\partial M \times \{0\} \subset \partial M \times [0,1]$ and extend X_t to a vector field (also denoted by X_t) on M_1, tangent to ∂M_1 on the boundary ∂M_1. Applying the previous theorem, and using the fact that the phase trajectories of X_t cross $\partial M \subset M_1$ in one direction, we obtain the following theorem.

THEOREM: Let M be a compact manifold with boundary, and $X_t \in TM$ a family of vector fields smoothly depending on $t \in [0,a]$. Suppose that $X_t|_{\partial M}$ points inside M. **Then there exists a unique family of smooth embeddings $V_t: M \longrightarrow M$, $t \in [0,a]$, such that $\frac{d}{dt}V_t^* = X_t$ and $V_0 = \mathrm{Id}$.** ■

Lie derivative and a flow of diffeomorphisms

DEFINITION: Let v_t be a vector field on M, smoothly depending on the "time parameter" $t \in [a,b]$, and $V: M \times [a,b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $\frac{d}{dt} V_t = v_t$ for each $t \in [a,b]$, and $V_0 = \operatorname{Id}$. Then V_t is called **an exponent of v_t**.

CLAIM: Exponent of a vector field is unique; it exists when M is compact. This statement is called **"Picard-Lindelöf theorem"** or **"uniqueness and existence of solutions of ordinary differential equations"**.

PROPOSITION: Let v_t be a time-dependent vector field, $t \in [a,b]$, and V_t its exponent. For any $\alpha \in \Lambda^* M$, consider $V_t^* \alpha$ as a $\Lambda^* M$-valued function of t. **Then $\operatorname{Lie}_{v_0}(\alpha) = \frac{d}{dt}|_{t=0}(V_t^* \alpha)$.**

Proof: By definition of differential, $\operatorname{Lie}_{v_0} f = \langle df, v_0 \rangle$, hence $\operatorname{Lie}_{v_0} f = \frac{d}{dt}|_{t=0} V_t^*(f)$. The operator Lie_{v_0} commutes with de Rham differential, because $\operatorname{Lie}_v = i_v d + d i_v$. The map $\frac{d}{dt} V_t$ commutes with de Rham differential, because it is a derivative of a pullback. Now **Lemma (*) is applied to show that $\operatorname{Lie}_{v_0} \alpha = \frac{d}{dt}|_{t=0}(V_t^* \alpha)$.** ■

Homotopy operators

DEFINITION: A complex is a sequence of vector spaces and homomorphisms ... $\xrightarrow{d} C_{i-1} \xrightarrow{d} C_i \xrightarrow{d} C_{i+1} \xrightarrow{d}$ ...
satisfying $d^2 = 0$. **Homomorphism** $(C_*, d) \longrightarrow (C'_*, d)$ of complexes is a sequence of homomorphism $C_i \longrightarrow C'_i$ commuting with the differentials.

DEFINITION: An element $c \in C_i$ is called **closed** if $c \in \ker d$ and **exact** if $c \in \operatorname{im} d$. **Cohomology** of a complex is a quotient $\frac{\ker d}{\operatorname{im} d}$.

REMARK: A homomorphism of complexes induces a natural homomorphism of cohomology groups.

DEFINITION: Let (C_*, d), (C'_*, d) be a complex. **Homotopy** is a sequence of maps $h: \ C_* \longrightarrow C'_{*-1}$. Two homomorphisms $f, g: \ (C_*, d) \longrightarrow (C'_*, d)$ are called **homotopy equivalent** if $f - g = \{h, d\}$ for some homotopy operator h.

CLAIM: Let $f, f': \ (C_*, d) \longrightarrow (C'_*, d)$ be homotopy equivalent maps of complexes. **Then f and f' induce the same maps on cohomology.**

Proof. Step 1: Let $g := f - f'$. It would suffice to prove that g induces 0 on cohomology.

Lie derivative and homotopy

CLAIM: Let $f, f' : (C_*, d) \longrightarrow (C'_*, d)$ be homotopy equivalent maps of complexes. **Then f and f' induce the same maps on cohomology.**

Proof. Step 1: Let $g := f - f'$. It would suffice to prove that g induces 0 on cohomology.

Step 2: Let $c \in C_i$ be a closed element. **Then $g(c) = dh(c) + hd(c) = dh(c)$ exact.** ■

REMARK: Let v be a vector field, and $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$ be the corresponding Lie derivative. Then **Lie_v commutes with the de Rham differential, and acts trivially on the de Rham cohomology.**

Proof: $\operatorname{Lie}_v = i_v d + d i_v$ maps closed forms to exact. ■

COROLLARY: Let V_t, $t \in [a, b]$ be a flow of diffeomorphisms on a manifold M. **Then V_b^* (the pullback) acts on cohomology the same way as V_a^*.**

Proof: Since $\frac{dV_t^*}{dt}(\eta) = \operatorname{Lie}_{X_t}(\eta)$, it acts trivially on cohomology. Then $V_b^* - V_a^*(\eta) = \int_a^b \operatorname{Lie}_{X_t}(\eta)$ is exact for any closed η. Therefore, $V_b^*(\eta) - V_a^*(\eta)$ is exact. ■