Topologia das Variedades

Cohomology, lecture 4: de Rham cohomology of a sphere

Misha Verbitsky

May 3, 2018

IMPA, Tuesdays and Thursdays, 10:30, Sala 224

The Grassmann algebra (reminder)

DEFINITION: Let V be a vector space, and $W \subset V \otimes V$ a subspace generated by vectors $x \otimes y + y \otimes x$ and $x \otimes x$, for all $x, y \in V$. A graded algebra defined by the generator space V and the relation space W is called **Grassmann algebra**, or **exterior algebra**, and denoted $\Lambda^*(V)$. The space $\Lambda^i(V)$ is called **i-th exterior power** of V, and the multiplication in $\Lambda^*(V)$ – **exterior multiplication**. Exterior multiplication is denoted $\wedge$.

EXERCISE: Prove that $\Lambda^1 V$ is isomorphic to V.

DEFINITION: An element of Grassmann algebra is called **even** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i}(V)$ and **odd** if it lies in $\bigoplus_{i\in\mathbb{Z}} \Lambda^{2i+1}(V)$. For an even or odd $x \in \Lambda^*(V)$, we define a number $\tilde{x}$ called **parity** of x. The parity of x is 0 for even x and 1 for odd.

CLAIM: In Grassmann algebra, $x \wedge y = (-1)^{\tilde{x}\tilde{y}} y \wedge x$.

De Rham algebra (reminder)

DEFINITION: Let $\alpha \in (V^*)^{\otimes i}$ and $\alpha \in (V^*)^{\otimes j}$ be polylinear forms on V. Define the **tensor multiplication** $\alpha \otimes \beta$ as

$$\alpha \otimes \beta(x_1, ..., x_{i+j}) := \alpha(x_1, ..., x_j)\beta(x_{i+1}, ..., x_{i+j}).$$

DEFINITION: Let $\bigotimes_k T^*M \xrightarrow{\Pi} \Lambda^k M$ be the antisymmetrization map,

$$\Pi(\alpha)(x_1, ..., x_n) := \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^\sigma \alpha(x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n}).$$

Define **the exterior multiplication** $\wedge : \Lambda^i M \times \Lambda^j M \longrightarrow \Lambda^{i+j} M$ as $\alpha \wedge \beta := \Pi(\alpha \otimes \beta)$, where $\alpha \otimes \beta$ is a section $\Lambda^i M \otimes \Lambda^j M \subset \bigotimes_{i+j} T^*M$ obtained as their tensor multiplication.

REMARK: The fiber of the bundle $\Lambda^* M$ at $x \in M$ **is identified with the Grassmann algebra $\Lambda^* T^*_x M$.** This identification is compatible with the Grassmann product.

DEFINITION: Let $t_1, ..., t_n$ be coordinate functions on $\mathbb{R}^n$, and $\alpha \in \Lambda^* \mathbb{R}^n$ a monomial obtained as a product of several dt_i: $\alpha = dt_{i_1} \wedge dt_{i_2} \wedge ... \wedge dt_{i_k}$ $i_1 < i_2 < ... < i_k$. Then α is called **a coordinate monomial**.

De Rham differential (reminder)

THEOREM: There exists a unique operator $C^\infty M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \Lambda^3 M \xrightarrow{d} ...$ satisfying the following properties

1. On functions, d is equal to the differential.
2. $d^2 = 0$
3. **(Graded Leibniz identity)** $d(\eta \wedge \xi) = d(\eta) \wedge \xi + (-1)^{\tilde\eta} \eta \wedge d(\xi)$, where $\tilde\eta = 0$ where $\eta \in \lambda^{2i} M$ is **an even form,** and $\eta \in \lambda^{2i+1} M$ is **odd.**

DEFINITION: The operator d is called **de Rham differential**.

DEFINITION: A form η is called **closed** if $d\eta = 0$, **exact** if $\eta \in \operatorname{im} d$. The group $\frac{\ker d}{\operatorname{im} d}$ is called **de Rham cohomology** of M.

Pullback of a differential form (reminder)

DEFINITION: Let $M \xrightarrow{\varphi} N$ be a morphism of smooth manifolds, and $\alpha \in \Lambda^i N$ be a differential form. Consider an i-form $\varphi^*\alpha$ taking value

$$\alpha\Big|_{\varphi(m)}(D_\varphi(x_1), ...D_\varphi(x_i))$$

on $x_1, ..., x_i \in T_m M$. It is called **the pullback of α**. If $M \xrightarrow{\varphi} N$ is a closed embedding, the form $\varphi^*\alpha$ is called **the restriction** of α to $M \hookrightarrow N$.

LEMMA: (*) Let $\Psi_1, \Psi_2 : \Lambda^* N \longrightarrow \Lambda^* M$ be two maps which satisfy graded Leibnitz identity, supercommutes with de Rham differential, and satisfy $\Psi_1|_{C^\infty M} = \Psi_2|_{C^\infty M}$. **Then $\Psi_1 = \Psi_2$.**

Proof: The algebra $\Lambda^* M$ is generated multiplicatively by $C^\infty M$ and $d(C^\infty M)$; restrictions of Ψ_i to these two spaces are equal. ∎

CLAIM: Pullback commutes with the de Rham differential.

Proof: Let $d_1, d_2 : \Lambda^* N \longrightarrow \Lambda^{*+1} M$ be the maps $d_1 = \varphi^* \circ d$ and $d_2 = d \circ \varphi^*$. **These maps satisfy the Leibnitz identity, and they are equal on $C^\infty M$.** The super-commutator $\delta := \{d_i, d\}$ is equal to $d \circ \varphi^* \circ d$, it commutes with d, and equal 0 on functions. By Lemma (*), $\delta = 0$. Then d_i supercommutes with d. Applying Lemma (*) again, we obtain that $d_1 = d_2$. ∎

Lie derivative (reminder)

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along** v if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra.

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v |_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

THEOREM: (Cartan's formula) Let i_v be a convolution with a vector field, $i_v(\eta) = \eta(v, \cdot, \cdot, ..., \cdot)$ **Then $\{d, i_v\}$ is equal to the Lie derivative along v.**

Flow of diffeomorphisms (reminder)

DEFINITION: Let $f: M \times [a,b] \longrightarrow M$ be a smooth map such that for all $t \in [a,b]$ the restriction $f_t := f\big|_{M\times \{t\}}: M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c}: C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^* \frac{dV_t}{dt}|_{t=c} f$. **Then $f \longrightarrow (V_t^{-1})^* \frac{d}{dt} V_t^* f$ is a derivation** (that is, a vector field).

THEOREM: Let M be a compact manifold, and $X_t \in TM$ a family of vector fields smoothly depending on $t \in [0,a]$. **Then there exists a unique diffeomorphism flow V_t, $t \in [0,a]$, such that $V_0 = \mathrm{Id}$ and $\frac{d}{dt}V_t^* = X_t$.**

Lie derivative and a flow of diffeomorphisms (reminder)

DEFINITION: Let v_t be a vector field on M, smoothly depending on the "time parameter" $t \in [a,b]$, and $V: M \times [a,b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $\frac{d}{dt}V_t = v_t$ for each $t \in [a,b]$, and $V_0 = \operatorname{Id}$. Then V_t is called **an exponent of v_t**.

CLAIM: Exponent of a vector field is unique; it exists when M is compact. This statement is called **"Picard-Lindelöf theorem"** or **"uniqueness and existence of solutions of ordinary differential equations"**.

PROPOSITION: Let v_t be a time-dependent vector field, $t \in [a,b]$, and V_t its exponent. For any $\alpha \in \Lambda^* M$, consider $V_t^* \alpha$ as a $\Lambda^* M$-valued function of t. **Then $\operatorname{Lie}_{v_t}(\alpha) = (V_t^{-1})^* \frac{d}{dt}(V_t^* \alpha)$.**

Lie derivative and cohomology

CLAIM: Let v be a vector field, and $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$ be the corresponding Lie derivative. Then **Lie_v commutes with the de Rham differential, and acts trivially on the de Rham cohomology.**

Proof: $\operatorname{Lie}_v = i_v d + d i_v$ maps closed forms to exact. ∎

COROLLARY: Let V_t, $t \in [a, b]$ be a flow of diffeomorphisms on a manifold M. **Then the pullback map V_b^* acts on cohomology the same way as V_a^*.**

Proof: Since $(V_t^{-1})^* \frac{dV_t^*}{dt}(\eta) = \operatorname{Lie}_{X_t}(\eta)$, this map it acts trivially on cohomology. Then $V_b^* - V_a^*(\eta) = \int_a^b V_t^* \operatorname{Lie}_{X_t}(\eta)$ is exact for any closed η. Therefore, $V_b^*(\eta) - V_a^*(\eta)$ is exact. ∎

Poincaré lemma

DEFINITION: An open subset $U \subset \mathbb{R}^n$ is called **starlike** if for any $x \in U$ the interval $[0, x]$ belongs to U.

THEOREM: (Poicaré lemma) Let $U \subset \mathbb{R}^n$ be a starlike subset. **Then $H^i(U) = 0$ for $i > 0$.**

Proof: Consider the map $V_t : U \longrightarrow U$ mapping x to tx, where $t \in [0, 1]$. Then the map $\eta \longrightarrow (V_t^{-1})^* \frac{dV_t}{dt}(\eta)$ commutes with d and maps closed forms to exact, and hence acts trivially on cohomology. **Therefore, $V_1^* - V_0^* = \int_0^1 (V_t^{-1})^* \frac{dV_t}{dt}$ also acts trivially on cohomology.** However, V_0^* maps any $\eta \in \Lambda^i(M)$, $i > 0$ to 0. ■

Homogeneous spaces

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\mathrm{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

THEOREM: **Homogeneous tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM,** for any $x \in M$.

■

Cohomology of homogeneous spaces

CLAIM: Let $M = G/H$ be a homogeneous space, with G connected and compact. Denote by $\Lambda^*(M)^G$ the space of G-invariant differential forms. Then **the natural morphism of complexes**

$$\begin{array}{ccccccc} \Lambda^0(M)^G & \stackrel{d}{\longrightarrow} & \Lambda^1(M)^G & \stackrel{d}{\longrightarrow} & \Lambda^2(M)^G & \stackrel{d}{\longrightarrow} & ... \\ \downarrow \iota & & \downarrow \iota & & \downarrow \iota & & \\ \Lambda^0(M) & \stackrel{d}{\longrightarrow} & \Lambda^1(M) & \stackrel{d}{\longrightarrow} & \Lambda^2(M) & \stackrel{d}{\longrightarrow} & ... \end{array}$$

induces an isomorphism on cohomology.

Proof: Consider the averaging map $\eta \stackrel{\mathsf{Av}}{\longrightarrow} \frac{\int_{g\in G} g^*(\eta) dg}{\mathsf{Vol}(G)}$ where the volume and the integral is taken with respect to a G-invariant measure dG on G. Since g^* commutes with the de Rham differential d, the map Av also commutes with d. Since $g^*(\eta)$ is cohomologous to η as shown above, the form $\mathsf{Av}(\eta)$ is cohomologous to η. Therefore, ι is invertible on cohomology: $\iota \circ \mathsf{Av} = \mathsf{Id}, \mathsf{Av} \circ \iota = \mathsf{Id}$. ■

Special orthogonal group $SO(n)$

DEFINITION: Orthogonal group $O(n)$ is the group of linear isometries of $\mathbb{R}^n$. **Special orthogonal group**, denoted $SO(n)$, is the group of orthogonal matrices $A \in \operatorname{End}(\mathbb{R}^n)$ of determinant 1.

REMARK: Clearly, $SO(n)$ **acts on $(n-1)$-dimensional sphere S^{n-1} transitively.**

The special orthogonal group $SO(n)$ is conected

THEOREM: The group $SO(n)$ is connected.

Proof: Each element $A \in SO(n)$ can be represented in a certain basis by a block matrix,

$$gAg^{-1} = \begin{pmatrix} A_{\alpha_1} & & & 0 \\ & A_{\alpha_2} & & \\ & & \ddots & \\ 0 & & & A_{\alpha_k} \end{pmatrix}$$

where $A_{\alpha_1} \in \operatorname{End}(\mathbb{R}^2)$ are rotation matrices,

$$A_{\alpha_i} = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i \\ -\sin\alpha_i & \cos\alpha_i \end{pmatrix}$$

Then

$$A_t := g^{-1} \begin{pmatrix} A_{t\alpha_1} & & & 0 \\ & A_{t\alpha_2} & & \\ & & \ddots & \\ 0 & & & A_{t\alpha_k} \end{pmatrix} g$$

is a homotopy connecting $A_t = A$ to $A_0 = \operatorname{Id}$. ■

COROLLARY: Cohomology of a sphere S^n are equal to the cohomology of the algebra of $SO(n+1)$-invariant differential forms.

Proof: We have just shown that $SO(n+1)$ is connected; its compactness is left as an exercise. ■

Special orthopgonal group $SO(n)$ acting on a sphere

THEOREM: Let α be an $SO(n+1)$-invariant p-form on S^n. **Then $p=0$ or $p=n$ and α is either a constant or a constant times the Riemannian volume form.**

Proof. Step 1: Using induction on n, we can assume that the statement of the theorem is true for S^k, $k<n$ (it is clearly true for S^1). Since $O(k+1)$ acts on S^k in non-orientable way, **all $O(k+1)$-invariant forms on S^k are constant 0-forms.**

Step 2: Let $S^k \subsetneq S^n$ be a sphere associated with a $k+1$-dimensional subspace $V=\mathbb{R}^{k+1}\subset W=\mathbb{R}^{n+1}$. Clearly, the stabilizer of S^k in $SO(n+1)$ acts on S^k as a full orthogonal group $O(k+1)$. **Therefore, any $SO(n+1)$-invariant form restricted to $S^k=S^n\cap V$ gives a constant 0-form,** for any $k+1$-dimensional subspace $V\subset W$.

Step 3: Suppose that there exists a non-zero $SO(n+1)$-invariant k-form α on S^n, with $0<k<n$. Then α restricted to S^k is 0, for any sphere $\mathfrak{S}=S^k\subset S^n$ obtained as $\mathfrak{S}=S^n\cap V$. However, there is such a sphere tangent to any k-dimensional subspace in T_xS^n for each $x\in S^n$. Therefore, $\alpha(v_1,...,v_k)=0$ for any $v_1,...,v_k\in T_xS^n$. ■

Cohomology of S^n

THEOREM: De Rham cohomology $H^p(S^n)$ of the sphere S^n are 1-dimensional for $p = 0, n$ and 0-dimensional otherwise.

Proof. Step 1: De Rham cohomology $H^0(X) = \mathbb{R}$ for any connected manifold M. Indeed, $\ker(d) \cap \Lambda^0(X)$ is the space of constant functions.

Step 2: De Rham cohomology $H^p(S^n)$ are represented by $SO(n+1)$-invariant forms, hence they vanish for $0 < p < n$. The space of $SO(n)$-invariant n-forms is 1-dimensional, because all such forms are proportional to the space of Riemannian volume forms. These forms are not cohomologous to 0 because the space $\Lambda^{n-1}(S^n)^{SO(n+1)}$ is trivial, and one can compute cohomology of S^n by taking the cohomology of the complex of $SO(n+1)$-invariant forms. ■