Topologia das Variedades

Cohomology, lecture 5: Mayer-Vietoris exact sequence

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Pullback of a differential form (reminder)

DEFINITION: Let $M \stackrel{\varphi}{\longrightarrow} N$ be a morphism of smooth manifolds, and $\alpha \in \Lambda^i N$ be a differential form. Consider an i-form $\varphi^*\alpha$ taking value

$$\alpha\Big|_{\varphi(m)}(D\varphi(x_1), ...D\varphi(x_i))$$

on $x_1, ..., x_i \in T_m M$. It is called **the pullback of α**. If $M \stackrel{\varphi}{\longrightarrow} N$ is a closed embedding, the form $\varphi^*\alpha$ is called **the restriction** of α to $M \hookrightarrow N$.

LEMMA: (*) Let $\Psi_1, \Psi_2 : \Lambda^* N \longrightarrow \Lambda^* M$ be two maps which satisfy graded Leibnitz identity, supercommutes with de Rham differential, and satisfy $\Psi_1|_{C^\infty M} = \Psi_2|_{C^\infty M}$. **Then $\Psi_1 = \Psi_2$.**

Proof: The algebra $\Lambda^* M$ is generated multiplicatively by $C^\infty M$ and $d(C^\infty M)$; restrictions of Ψ_i to these two spaces are equal. ■

CLAIM: Pullback commutes with the de Rham differential.

Proof: Let $d_1, d_2 : \Lambda^* N \longrightarrow \Lambda^{*+1} M$ be the maps $d_1 = \varphi^* \circ d$ and $d_2 = d \circ \varphi^*$. **These maps satisfy the Leibnitz identity, and they are equal on $C^\infty M$.** The super-commutator $\delta := \{d_i, d\}$ is equal to $d \circ \varphi^* \circ d$, it commutes with d, and equal 0 on functions. By Lemma (*), $\delta = 0$. Then d_i supercommutes with d. Applying Lemma (*) again, we obtain that $d_1 = d_2$. ■

Lie derivative (reminder)

DEFINITION: Let B be a smooth manifold, and $v \in TM$ a vector field. An endomorphism $\operatorname{Lie}_v: \Lambda^* M \longrightarrow \Lambda^* M$, preserving the grading is called **a Lie derivative along v** if it satisfies the following conditions.

(1) On functions Lie_v is equal to a derivative along v. (2) $[\operatorname{Lie}_v, d] = 0$.

(3) Lie_v is a derivation of the de Rham algebra.

REMARK: The algebra $\Lambda^*(M)$ is generated by $C^\infty M = \Lambda^0(M)$ and $d(C^\infty M)$. The restriction $\operatorname{Lie}_v \big|_{C^\infty M}$ is determined by the first axiom. On $d(C^\infty M)$ is also determined because $\operatorname{Lie}_v(df) = d(\operatorname{Lie}_v f)$. **Therefore, Lie_v is uniquely defined by these axioms.**

THEOREM: (Cartan's formula) Let i_v be a convolution with a vector field, $i_v(\eta) = \eta(v, \cdot, \cdot, ..., \cdot)$ **Then $\{d, i_v\}$ is equal to the Lie derivative along v.**

Flow of diffeomorphisms (reminder)

DEFINITION: Let $f: M \times [a,b] \longrightarrow M$ be a smooth map such that for all $t \in [a,b]$ the restriction $f_t := f\Big|_{M\times\{t\}}: M \longrightarrow M$ is a diffeomorphism. Then f is called **a flow of diffeomorphisms**.

CLAIM: Let V_t be a flow of diffeomorphisms, $f \in C^\infty M$, and $V_t^*(f)(x) := f(V_t(x))$. Consider the map $\frac{d}{dt}V_t|_{t=c}: C^\infty M \longrightarrow C^\infty M$, with $\frac{d}{dt}V_t|_{t=c}(f) = (V_c^{-1})^* \frac{dV_t}{dt}|_{t=c} f$. **Then $f \longrightarrow (V_t^{-1})^*\frac{d}{dt}V_t^* f$ is a derivation** (that is, a vector field).

THEOREM: Let M be a compact manifold, and $X_t \in TM$ a family of vector fields smoothly depending on $t \in [0,a]$. **Then there exists a unique diffeomorphism flow V_t, $t \in [0,a]$, such that $V_0 = \operatorname{Id}$ and $\frac{d}{dt}V_t^* = X_t$.**

Lie derivative and a flow of diffeomorphisms (reminder)

DEFINITION: Let v_t be a vector field on M, smoothly depending on the "time parameter" $t \in [a,b]$, and $V: M \times [a,b] \longrightarrow M$ a flow of diffeomorphisms which satisfies $\frac{d}{dt}V_t = v_t$ for each $t \in [a,b]$, and $V_0 = \operatorname{Id}$. Then V_t is called **an exponent of v_t**.

CLAIM: Exponent of a vector field is unique; it exists when M is compact. This statement is called **"Picard-Lindelöf theorem"** or **"uniqueness and existence of solutions of ordinary differential equations"**.

PROPOSITION: Let v_t be a time-dependent vector field, $t \in [a,b]$, and V_t its exponent. For any $\alpha \in \Lambda^* M$, consider $V_t^* \alpha$ as a $\Lambda^* M$-valued function of t. **Then** $\operatorname{Lie}_{v_t}(\alpha) = (V_t^{-1})^* \frac{d}{dt}(V_t^* \alpha)$.

Lie derivative and cohomology (reminder)

CLAIM: Let v be a vector field, and $\operatorname{Lie}_v : \Lambda^* M \longrightarrow \Lambda^* M$ be the corresponding Lie derivative. Then **Lie_v commutes with the de Rham differential, and acts trivially on the de Rham cohomology.**

Proof: $\operatorname{Lie}_v = i_v d + d i_v$ maps closed forms to exact. ■

COROLLARY: Let V_t, $t \in [a,b]$ be a flow of diffeomorphisms on a manifold M. **Then the pullback map V_b^* acts on cohomology the same way as V_a^*.**

Proof: Since $(V_t^{-1})^* \frac{dV_t^*}{dt}(\eta) = \operatorname{Lie}_{X_t}(\eta)$, this map it acts trivially on cohomology. Then $V_b^* - V_a^*(\eta) = \int_a^b V_t^* \operatorname{Lie}_{X_t}(\eta)$ is exact for any closed η. Therefore, $V_b^*(\eta) - V_a^*(\eta)$ is exact. ■

Extension of vector fields

EXERCISE: Let $M \subset M_1$ be a smooth closed submanifold and $v \in TM_1|_M$ a tangent to M_1 vector field defined on M. **Prove that v can be extended to a vector field on M_1.** Moreover, this extension can be chosen smoothly for any smooth family of vector fields v.

Homotopy equivalence of cohomology

THEOREM: Let $F_t: X \longrightarrow Y$ be morphisms (smooth maps) of compact manifolds, smoothly depending on $t \in [a,b]$. Consider the corresponding maps on cohomology: $F_t^*: H^*(Y) \longrightarrow H^*(X)$. **Then $F_a^* = F_b^*$.**

Proof. Step 1: Consider the graph $\Gamma_t \subset X \times Y$ of F_t. Then Γ_t is a smooth submanifold in $X \times Y$, and $v_x := \frac{d}{dt} F_t(x) \in T_{F_t(x)} Y$ gives a tangent vector $(0, v_x) \in T_{x, F_t(x)} X \times Y = T_x Y \times T_{F_t(x)} X$. Taking all v_x for all points $(x, F_t(x)) \in \Gamma_t$, we obtain a vector field $\beta_t \in T(X \times Y)\Big|_{\Gamma_t}$.

Step 2: Using the previous exercise, we extend β_t to a vector field on $X \times Y$. The corresponding flow of diffeomorphisms $V_t: X \times Y \longrightarrow X \times Y$ maps $(x, F_a(x))$ to $(x, F_t(x))$.

Step 3: Since the flows of diffeomorphisms induce constant maps on cohomology, the natural restriction maps $H^*(X \times Y) \longrightarrow H^*(\Gamma_a) = H^*(X)$ and $H^*(X \times Y) \longrightarrow H^*(\Gamma_b) = H^*(X)$ are equal. However, the pullback map $F_t^*: H^*(Y) \longrightarrow H^*(X)$ can be obtained as a composition of the pulback map $H^*(Y) \longrightarrow H^*(X \times Y)$ associated with the projection, and the restriction $H^*(X \times Y) \longrightarrow H^*(\Gamma_t) = H^*(X)$. Therefore $F_a^* = F_b^*$, and the de Rham cohomology are homotopy invariant. ■

Tuesday lecture
is moved to May 14, Monday, 10:30, room 228

Brouwer fixed point theorem: a preparatory lemma

LEMMA: Let $B \subset \mathbb{R}^n$ be a closed ball, and $\Psi: B \longrightarrow \partial B$ be a map to its boundary ∂B such that $\Psi|_{\partial B} = \mathrm{Id}_{\partial B}$. **Then Ψ cannot be smooth.**

Proof: Suppose that Ψ is smooth. Consider the composition $j \circ \Psi$ of embedding $j: \partial B \longrightarrow B$ and Ψ. Since $H^{n-1}(S^{n-1}) = \mathbb{R}$ and $j \circ \Psi = \mathrm{Id}_{\partial B}$, the map $\Psi^* \circ j^*$ acts as identity on $H^{n-1}(S^{n-1})$. This is impossible because **by Poincaré lemma $H^{n-1}(B) = 0$, hence $j^*: H^{n-1}(B) \longrightarrow H^{n-1}(S^{n-1})$ is zero, and composition of Ψ^* and j^* vanishes on $H^{n-1}(S^{n-1})$.** ∎

Brouwer fixed point theorem

THEOREM: (Brouwer fixed point theorem)
Let $B \subset \mathbb{R}^n$ be a closed ball, and $\varphi: B \longrightarrow B$ be a continuous map. **Then B has at least one fixed point.**

Proof. Step 1: Suppose that φ has no fixed points. Since B is compact, **there exists $\varepsilon > 0$ such that $d(x, \varphi(x)) > 2\varepsilon$ for each $x \in B$.**

Step 2: By Stone-Weierstrass, any continuous map can be ε-approximated by a smooth map φ_1 such that $d(x, \varphi_1(x)) \geqslant d(x, \varphi(x)) - d(\varphi(x), \varphi_1(x)) > \varepsilon$. This implies that φ_1 has no fixed points, and **we may assume that φ is smooth.**

Step 3: Let $l_x \subset \mathbb{R}^n$ be an oriented line connecting x to $\varphi(x)$. Then l_x intersects the boundary ∂B in two points, distinguished by its orientation. Let $\Psi(x)$ be the first point of intersection of l_x and ∂B. Clearly, **the map $x \longrightarrow \Psi(x)$ is smooth and $\Psi(x) = x$ for all $x \in \partial B$.** This is impossible by the previous lemma. ■

Long exact sequence

DEFINITION: A complex is a sequence of vector spaces and homomorphisms ... $\xrightarrow{d} C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \xrightarrow{d}$ ... satisfying $d^2 = 0$. **Homomorphism** $(C^*, d) \longrightarrow (C_1^*, d)$ of complexes is a sequence of homomorphism $C^i \longrightarrow C_1^i$ commuting with the differentials.

DEFINITION: An element $c \in C^i$ is called **closed** if $c \in \ker d$ and **exact** if $c \in \operatorname{im} d$. **Cohomology** of a complex is a quotient $\frac{\ker d}{\operatorname{im} d}$. One denotes the i-th group of cohomology of a complex by $H^i(C^*)$

REMARK: A homomorphism of complexes induces a natural homomorphism of cohomology groups.

DEFINITION: Short exact sequence of complexes $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ is a sequence of morphisms of complexes $A^* \xrightarrow{x} B^* \xrightarrow{y} C^*$ such that $x: A^i \longrightarrow B^i$ is injective, $y: B^i \longrightarrow C^i$ is surjective (for all i), and $\ker y = \operatorname{im} x$.

THEOREM: Let $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ be an exact sequence of complexes. Then there exists a **long exact sequence of cohomology**

$$... \longrightarrow H^{i-1}(C^*) \longrightarrow H^i(A^*) \longrightarrow H^i(B^*) \longrightarrow H^i(C^*) \longrightarrow H^{i+1}(A^*) \longrightarrow ...$$

Differential forms on closed subsets

DEFINITION: Let M be a manifold, and $Z \subset M$ a closed subset. Let α, β be two differential forms defined in open sets U_α and U_β containing Z. We say that α and β are equivalent if $\alpha = \beta$ on $U_\alpha \cap U_\beta$. The space of equivalence classes is denoted $\Lambda^*(Z)$ and called **the space of differential forms on Z** or **the space of germs of differentials forms in Z**.

REMARK: If Z is a manifold with boundary, $\Lambda^*(Z)$ is the space of differential forms on Z, by definition of differential forms on manifolds with boundary.

EXERCISE: Using partition of unity, **prove that the natural restriction map $\Lambda^*(M) \longrightarrow \Lambda^*(Z)$ is surjective for any closed $Z \subset M$.**

REMARK: The de Rham differential is well defined on $\Lambda^*(Z)$, allowing us to **define the de Rham cohomology of Z as usual.** Poincaré lemma holds in this situation, too.

CLAIM: Let $X \subset \mathbb{R}^n$ be a closed starlike subset. **Then $H^i(X) = 0$ for all $i > 0$.**

Proof: Same as for open X. ■

Mayer-Vietoris long exact sequence

CLAIM: Let $X, Y \subset M$ be closed subsets. Then **the restriction maps define an exact sequence of complexes:**

$$0 \longrightarrow \Lambda^*(X \cup Y) \xrightarrow{\varphi} \Lambda^*(X) \oplus \Lambda^*(Y) \xrightarrow{\psi} \Lambda^*(X \cap Y) \longrightarrow 0. \quad (*)$$

Here φ is restriction to both components, and ψ is restriction $|_{X\cap Y}$ on the first component and $-|_{X\cap Y}$ on the second component.

Proof: The map φ is clearly injective; ψ is surjective because all forms in $\Lambda^*(X\cap Y)$ can be smoothly extended to M. Now, $\ker\psi$ is pairs $\alpha \in \Lambda^*(X), \beta \in \Lambda^*(Y)$ such that $\alpha|_{X\cap Y} = -\beta|_{X\cap Y}$, and this is precisely pairs which agree on an open neighbourhood of $X\cap Y$, that is, obtained by restriction from some $\gamma \in \Lambda^*(X\cup Y)$. ■

COROLLARY: (Mayer-Vietoris long exact sequence) Let $X, Y \subset M$ be closed subsets. **Then there is a long exact sequence associated with (*):**

$$\begin{aligned} ... \longrightarrow H^{i-1}(X) \oplus H^{i-1}(Y) \longrightarrow H^{i-1}(X\cap Y) \longrightarrow H^i(X \cup Y) \longrightarrow \\ \longrightarrow H^i(X) \oplus H^i(Y) \longrightarrow H^i(X\cap Y) \longrightarrow H^{i+1}(X\cup Y) \longrightarrow ... \end{aligned}$$

■

Computation of de Rham cohomology

THEOREM: Let M be a smooth manifold. **Then M admits a polyhedral structure and a triangulation.**

Proof: Was proven earier. ■

COROLLARY: Let M be a compact smooth manifold. **Then the cohomology of M are finite-dimensional.**

Proof. Step 1: Suppose that M is a union of n closed subsets K_1, ..., K_n which are starlike in some coordinates, and such that any intersection of any family of K_i is also starlike. Using induction by n, we may assume that the union of $n-1$ starlike sets with this property has finite cohomology. Consider the Mayer-Vietoris exact sequence

$$... \longrightarrow H^{i-1}(X \cap Y) \longrightarrow H^i(X \cup Y) \longrightarrow H^i(X) \oplus H^i(Y) \longrightarrow ...$$

where $X = K_1$ and $Y = \bigcup_{i=2}^n K_i$. Induction assumption gives that $H^i(Y)$ and $H^i(X \cap Y)$ has finite-dimensional cohomology, and $H^i(X)$ is finite-dimensional by Poincaré lemma. Then $H^i(X \cup Y)$ is also finite-dimensional.

Step 2: For any triangulation or a polyhedral structure, the simplices or polyhedra and their intersections are starlike, hence satisfy assumptions of Step 1. ■

Singular cohomology

DEFINITION: Let M be a topological space and C_i **the group of i-chains,** that is, the group freely generated by continuous maps $f: \Delta^i \longrightarrow M$, where Δ^i is i-simplex. **Boundary operator** maps a k-simplex f to a sum $\sum_{i=0}^k (-1)^k f_i$, where $f_i \in C_{i-1}$ is i-the face of f. Cohomology of this complex are called **singular homology** of M.

DEFINITION: The group of i-cochains is $C^i := \operatorname{Hom}(C_i, \mathbb{Z})$. The boundary operator $\partial: C_i \longrightarrow C_{i-1}$ defines **the coboundary operator** (denoted by the same letter) $\partial: C^i \longrightarrow C^{i-1}$. The cohomology of the complex

$$... \xrightarrow{\partial} C^i \xrightarrow{\partial} C^{i+1} \xrightarrow{\partial} ...$$

are called **singular cohomology of M**.

DEFINITION: Let R be a ring (typically, R will be a field $\mathbb{R}$). **The group of i-cochains with values in R** is $C^i_R := \operatorname{Hom}_{\mathbb{Z}}(C_i, R)$, where $\operatorname{Hom}_{\mathbb{Z}}$ denotes the homomorphism of abelian groups. Cohomology of the corresponding cochain complex are called **cohomology with coefficients in R**.

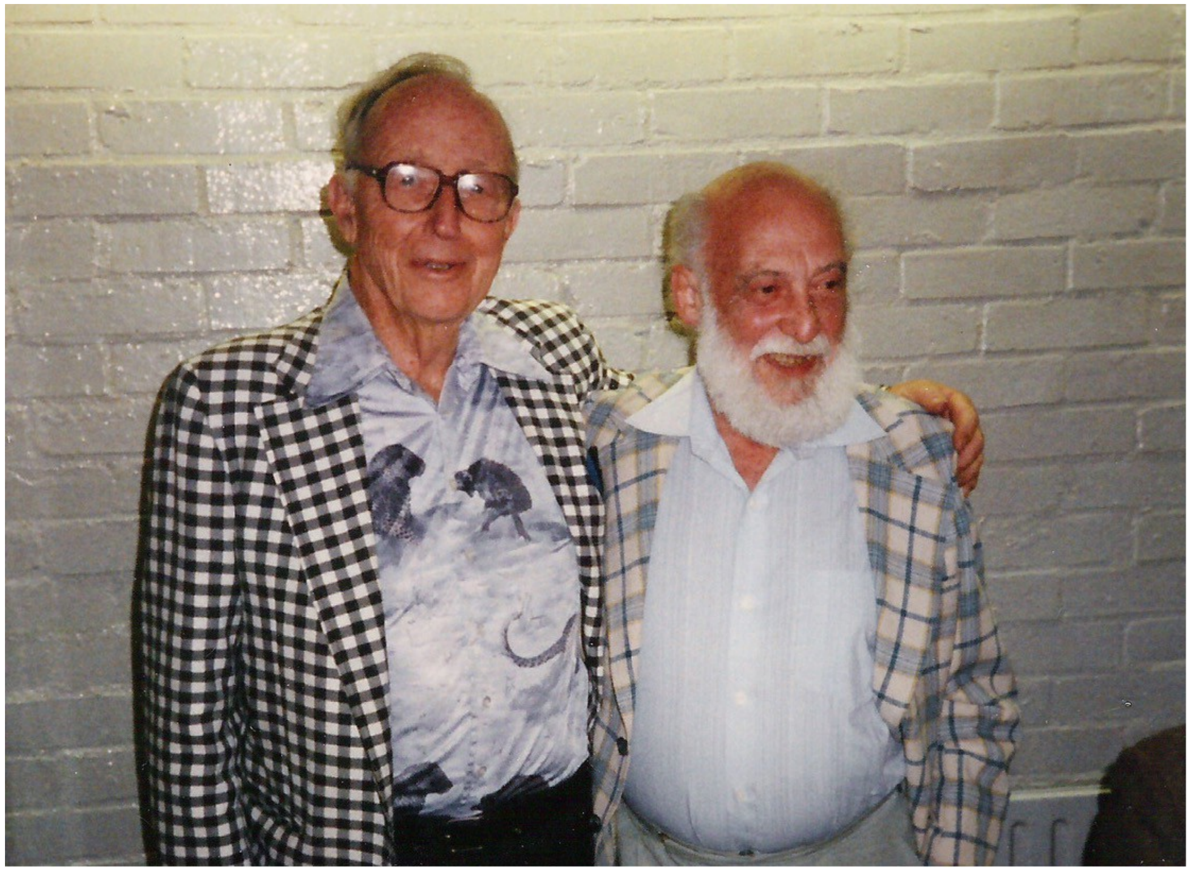
Singular cohomology

In the next lecture, I will prove

THEOREM: (de Rham theorem) Let $H^i_{dR}(M)$ denote de Rham cohomology of a manifold M and $H^i(M,\mathbb{R})$ denote the singular cohomology with coefficients in $\mathbb{R}$. **Then $H^i_{dR}(M) = H^i(M,\mathbb{R})$.**

To prove equivalence of different cohomology theories,

Eilenberg and MacLane developed the theory of categories.



Saunders MacLane (1909-2005) and Samuel Eilenberg (1913-1998)

Categories

DEFINITION: A category $\mathcal{C}$ is a collection of data called "objects" and "morphisms between objects" which satisfies the axioms below.

DATA.

Objects: A class $\mathcal{O}b(\mathcal{C})$ of **objects** of $\mathcal{C}$.

Morphisms: For each $X, Y \in \mathcal{O}b(\mathcal{C})$, one has a set $\mathcal{M}or(X, Y)$ of **morphisms from X to Y**.

Composition of morphisms: For each $\varphi \in \mathcal{M}or(X, Y), \psi \in \mathcal{M}or(Y, Z)$ there exists **the composition** $\varphi \circ \psi \in \mathcal{M}or(X, Z)$

Identity morphism: For each $A \in \mathcal{O}b(\mathcal{C})$ there exists a morphism $\operatorname{Id}_A \in \mathcal{M}or(A, A)$.

AXIOMS.

Associativity of composition: $\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$.

Properties of identity morphism: For each $\varphi \in \mathcal{M}or(X, Y)$, one has $\operatorname{Id}_x \circ \varphi = \varphi = \varphi \circ \operatorname{Id}_Y$

Categories (2)

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ be7objects of $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}or(X, Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y, X)$ such that $\varphi \circ \psi = \operatorname{Id}_X$ and $\psi \circ \varphi = \operatorname{Id}_Y$. In this case, the objects X and Y are called **isomorphic**.

Examples of categories:
Category of sets: its morphisms are arbitrary maps.
Category of vector spaces: its morphisms are linear maps.
Categories of rings, groups, fields: morphisms are homomorphisms.
Category of topological spaces: morphisms are continuous maps.
Category of smooth manifolds: morphisms are smooth maps.

Functors

DEFINITION: Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories. A **covariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the following set of data.

1. **A map** $F: \mathcal{O}b(\mathcal{C}_1) \longrightarrow \mathcal{O}b(\mathcal{C}_2)$.

2. **A map** $F: \mathcal{M}or(X,Y) \longrightarrow \mathcal{M}or(F(X),F(Y))$ **defined for any pair of objects** $X, Y \in \mathcal{O}b(\mathcal{C}_1)$**.**

These data define a functor if they are **compatible with compositions,** that is, satisfy $F(\varphi) \circ F(\psi) = F(\varphi \circ \psi)$ for any $\varphi \in \mathcal{M}or(X,Y)$ and $\psi \in \mathcal{M}or(Y,Z)$, and **map identity morphism to identity** morphism.

Example of functors

A "natural operation" on mathematical objects is usually a functor. Examples:

1. A map $X \longrightarrow 2^X$ from the set X to the set of all subsets of X is a functor from the category $\mathcal{S}ets$ of sets to itself.

2. A map $M \longrightarrow M^2$ mapping a topological space to its product with itself is a functor on topological spaces.

3. A map $V \longrightarrow V \oplus V$ is a functor on vector spaces; same for a map $V \longrightarrow V \otimes V$ or $V \longrightarrow (V \oplus V) \otimes V$.

4. **Identity functor** from any category to itself.

5. A map from topological spaces to $\mathcal{S}ets$, putting a topological space to the set of its connected components.

EXERCISE: Prove that it is a functor.

Contravariant functors

DEFINITION: Let $\mathcal{C}$ be a category. Define the **opposite category** $\mathcal{C}^{op}$ with the same set of objects, and $\mathcal{M}or_{\mathcal{C}^{op}}(A,B) = \mathcal{M}or_{\mathcal{C}}(B,A)$. The composition $\varphi \circ \psi$ in $\mathcal{C}$ gives the composition $\psi^{op} \circ \varphi^{op}$ in $\mathcal{C}^{op}$.

DEFINITION: A **contravariant functor** from $\mathcal{C}_1$ to $\mathcal{C}_2$ is the usual ("covariant") functor from $\mathcal{C}_1$ to $\mathcal{C}_2^{op}$.

EXAMPLE: A map from the category of topological spaces to category of rings mapping a space to a ring of continuous functions on it gives a contravariant functor.

EXAMPLE: Let $X \in \mathcal{O}b(\mathcal{C})$ be an object of $\mathcal{C}$. A map $Y \longrightarrow \mathcal{M}or(X,Y)$ defines a covariant functor from $\mathcal{C}$ to the category $\mathcal{S}ets$ of sets. A map $Y \longrightarrow \mathcal{M}or(Y,X)$ defines a contravariant functor from $\mathcal{C}$ to $\mathcal{S}ets$. Such functors to $\mathcal{S}ets$ are called **representable**.

Equivalence of functors

DEFINITION: Let $X, Y \in \mathcal{O}b(\mathcal{C})$ be objects of a category $\mathcal{C}$. A morphism $\varphi \in \mathcal{M}or(X,Y)$ is called **an isomorphism** if there exists $\psi \in \mathcal{M}or(Y,X)$ such that $\varphi\circ\psi = \operatorname{Id}_X$ and $\psi\circ\varphi = \operatorname{Id}_Y$. In this case X and Y are called **isomorphic**.

DEFINITION: Two functors $F, G: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ are called **equivalent** if for any $X \in \mathcal{O}b(\mathcal{C}_1)$ we are given an isomorphism $\Psi_X: F(X) \longrightarrow G(X)$, in such a way that for any $\varphi \in \mathcal{M}or(X,Y)$, one has $F(\varphi)\circ\Psi_Y = \Psi_X \circ G(\varphi)$.

REMARK: Such commutation relations are usually expressed by **commutative diagrams**. For example, the condition $F(\varphi)\circ\Psi_Y = \Psi_X \circ G(\varphi)$ is expressed by a commutative diagram

$$
\begin{array}{ccc}
F(X) & \xrightarrow{F(\varphi)} & F(Y) \\
{\scriptstyle \Psi_X}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_Y} \\
G(X) & \xrightarrow{G(\varphi)} & G(Y)
\end{array}
$$

A collection of morphism $\Psi_X: F(X) \longrightarrow G(X)$ is called **a natural transform of functors** if this diagram is commutative for all $\varphi \in \mathcal{M}or(X,Y)$.

Equivalence of cohomology: framework argument

REMARK: All cohomology theories give a contravariant functor from topological spaces to abelian groups.

THEOREM: Let H_1^i, H_2^i be contravariant functors ("cohomology theories") from the category of topological spaces to vector spaces, where $i = 0, 1, 2, ...$ Suppose that they satisfy the following axioms.

1. For any starlike set B, one has $H_1^i(B) = H_2^i(B) = 0$ and $H_i^0(B) = \mathbb{R}$.

2. Both cohomology theories H_1^* and H_2^* have Mayer-Vietoris long exact sequences.

3. There is a natural transform of functors $H_1 \xrightarrow{\Psi} H_2$ inducing isomorphism on cohomology of a starlike set.

Then Ψ is an equivalence of functors.

In this situation we also say that **cohomology theories H_1^* and H_2^* are equivalent.**