Topologia das Variedades

Cohomology, lecture 6: de Rham theorem

Misha Verbitsky

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Long exact sequence (reminder)

DEFINITION: A complex is a sequence of vector spaces and homomorphisms $... \xrightarrow{d} C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \xrightarrow{d} ...$ satisfying $d^2 = 0$. **Homomorphism** $(C^*, d) \longrightarrow (C_1^*, d)$ of complexes is a sequence of homomorphism $C^i \longrightarrow C_1^i$ commuting with the differentials.

DEFINITION: An element $c \in C^i$ is called **closed** if $c \in \ker d$ and **exact** if $c \in \operatorname{im} d$. **Cohomology** of a complex is a quotient $\frac{\ker d}{\operatorname{im} d}$. One denotes the i-th group of cohomology of a complex by $H^i(C^*)$

REMARK: A homomorphism of complexes induces a natural homomorphism of cohomology groups.

DEFINITION: Short exact sequence of complexes
$0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ is a sequence of morphisms of complexes $A^* \xrightarrow{x} B^* \xrightarrow{y} C^*$ such that $x: A^i \longrightarrow B^i$ is injective, $y: B^i \longrightarrow C^i$ is surjective (for all i), and $\ker y = \operatorname{im} x$.

THEOREM: Let $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ be an exact sequence of complexes. Then there exists a **long exact sequence of cohomology**

$$... \longrightarrow H^{i-1}(C^*) \longrightarrow H^i(A^*) \longrightarrow H^i(B^*) \longrightarrow H^i(C^*) \longrightarrow H^{i+1}(A^*) \longrightarrow ...$$

Differential forms on closed subsets (reminder)

DEFINITION: Let M be a manifold, and $Z \subset M$ a closed subset. Let α, β be two differential forms defined in open sets U_α and U_β containing Z. We say that α and β are equivalent if $\alpha = \beta$ on $U_\alpha \cap U_\beta$. The space of equivalence classes is denoted $\Lambda^*(Z)$ and called **the space of differential forms on Z** or **the space of germs of differentials forms in Z**.

REMARK: If Z is a manifold with boundary, $\Lambda^*(Z)$ is the space of differential forms on Z, by definition of differential forms on manifolds with boundary.

EXERCISE: Using partition of unity, **prove that the natural restriction map $\Lambda^*(M) \longrightarrow \Lambda^*(Z)$ is surjective for any closed $Z \subset M$.**

REMARK: The de Rham differential is well defined on $\Lambda^*(Z)$, allowing us to **define the de Rham cohomology of Z as usual.** Poincaré lemma holds in this situation, too.

CLAIM: Let $X \subset \mathbb{R}^n$ be a closed starlike subset. **Then $H^i(X) = 0$ for all $i > 0$.**

Proof: Same as for open X. ■

Mayer-Vietoris long exact sequence (reminder)

CLAIM: Let $X, Y \subset M$ be closed subsets. Then **the restriction maps define an exact sequence of complexes:**

$$0 \longrightarrow \Lambda^*(X \cup Y) \xrightarrow{\varphi} \Lambda^*(X) \oplus \Lambda^*(Y) \xrightarrow{\psi} \Lambda^*(X \cap Y) \longrightarrow 0. \quad (*)$$

Here φ is restriction to both components, and ψ is restriction $|_{X\cap Y}$ on the first component and $-|_{X\cap Y}$ on the second component.

Proof: The map φ is clearly injective; ψ is surjective because all forms in $\Lambda^*(X\cap Y)$ can be smoothly extended to M. Now, $\ker\psi$ is pairs $\alpha \in \Lambda^*(X), \beta \in \Lambda^*(Y)$ such that $\alpha|_{X\cap Y} = -\beta|_{X\cap Y}$, and this is precisely pairs which agree on an open neighbourhood of $X \cap Y$, that is, obtained by restriction from some $\gamma \in \Lambda^*(X \cup Y)$. ■

COROLLARY: **(Mayer-Vietoris long exact sequence)** Let $X, Y \subset M$ be closed subsets. **Then there is a long exact sequence associated with (*):**

$$\begin{aligned} ... \longrightarrow H^{i-1}(X) \oplus H^{i-1}(Y) \longrightarrow H^{i-1}(X\cap Y) \longrightarrow H^i(X \cup Y) \longrightarrow \\ \longrightarrow H^i(X) \oplus H^i(Y) \longrightarrow H^i(X\cap Y) \longrightarrow H^{i+1}(X \cup Y) \longrightarrow ... \end{aligned}$$

■

Singular cohomology

DEFINITION: Let M be a topological space and C_i **the group of singular i-chains,** that is, the group freely generated by continuous maps $f: \Delta^k \longrightarrow M$, where Δ^k is i-simplex. **Boundary operator** maps a k-simplex f to a sum $\sum_{i=0}^k (-1)^i f_i$, where $f_i \in C_{i-1}$ is i-th face of f. Cohomology of this complex are called **singular homology** of M.

DEFINITION: The group of i-cochains is $C^i := \operatorname{Hom}(C_i, \mathbb{Z})$. The boundary operator $\partial: C_i \longrightarrow C_{i-1}$ defines **the coboundary operator** (denoted by the same letter) $\partial: C^i \longrightarrow C^{i-1}$. The cohomology of the complex

$$... \xrightarrow{\partial} C^i \xrightarrow{\partial} C^{i+1} \xrightarrow{\partial} ...$$

are called **singular cohomology of M.**

DEFINITION: Let R be a ring (typically, R will be a field $\mathbb{R}$). **The group of i-cochains with values in R** is $C^i_R := \operatorname{Hom}_{\mathbb{Z}}(C_i, R)$, where $\operatorname{Hom}_{\mathbb{Z}}$ denotes the homomorphism of abelian groups. Cohomology of the corresponding cochain complex are called **cohomology with coefficients in R.**

Properties of singular cohomology

REMARK: Singular homology are functorial. Indeed, for any continuous map $f : X \longrightarrow Y$, the map f maps any chain in X to a chain in Y. Then any cochain on Y gives a cochain on X. This operation is called **the pullback** of a cochain.

CLAIM: **Singular cohomology are homotopy invariant,** that is, for any map $F_t : X \times [0,1] \longrightarrow Y$, the map $F_0 : X \longrightarrow Y$, $F_0(x) = F_t(x \times \{0\})$ acts on the cohomology in the same way as $F_1 : X \longrightarrow Y$, $F_1(x) = F_t(x \times \{1\})$.

Proof: Left as an exercise for now (we shall return to it later). ■

Homology are dual to cohomology

DEFINITION: Define **singular chain with coefficients in a field R** as a formal linear combination of maps $f: \Delta^k \longrightarrow M$, with coefficients in R. The boundary map ∂ is defined the space $C_k(R)$ of such chains in a natural way. **homology with coefficients in R** is $\frac{\ker \partial|_{C_k(R)}}{\partial(C_{k+1}(R))}$.

REMARK: There is a natural pairing between homology and cohomology. Indeed, a coboundary vanishes on cycles, hence the natural pairing between a cycle and a cocycle descends to cohomologies.

DEFINITION: A non-degenerate, or **perfect** pairing between two vector spaces V and W over a field R is a bilinear map $B: V \times W \longrightarrow R$ such that for any non-zero vector $v \in V$ there exists $w \in W$ such that $B(v,w) \neq 0$, and for all non-zero $w \in W$ there exists $v \in V$ with such a property.

THEOREM: Let R be a field. **Then the natural pairing $H_i(M,R) \times H^i(M,R) \longrightarrow R$ is non-degenerate,** and, moreover, $H^i(M,R) = H_i(M,R)^*$.

Proof: see the next slide

Homology are dual to cohomology (2)

THEOREM: Let R be a field. **Then the natural pairing $H_i(M,R) \times H^i(M,R) \longrightarrow R$ is non-degenerate,** and, moreover, $H^i(M,R) = H_i(M,R)^*$.

Proof. Step 1: Let $A: V \longrightarrow W$ be a linear map, and $A^*: W^* \longrightarrow V^*$ the dual map. Then the natural pairing between $\ker A$ and $V^*/\operatorname{im} A^*$ and $\operatorname{im} A$ and $W^*/\ker A^*$ is non-degenerate, and, moreover, $(\operatorname{im} A)^* = W^*/\ker A^*$. Indeed, $x \in \ker A \Leftrightarrow \langle Ax, y\rangle = 0 \Leftrightarrow \langle x, A^*y\rangle$, hence $\operatorname{im} A^*$ is precisely the space of functionals vanishing on $\ker A$. Similarly, $\langle A(z), y\rangle = 0 \Leftrightarrow \langle z, A^*(y)\rangle$, hence one $y \in W^*$ vanishes on $\operatorname{im} A$ if and only if $A^*(y) = 0$.

Step 2: Let $d: C \longrightarrow C$ be a map which satisfies $d^2 = 0$. Step 1 gives an isomorphism $(\ker d)^* = C^*/\operatorname{im} d^*$ and $(\operatorname{im} d)^* = C^*/\ker d^*$. Then $(\ker d/\operatorname{im} d)^* = \frac{\ker d^*}{\operatorname{im} d^*}$. ■

REMARK: This statement makes no sense when the coefficients are not a field. In this case **the universal coefficients theorem** gives a precise formula relating homology and cohomology.

Cohomology of smooth chains

PROPOSITION: Let C_k be the space of k-chains on a smooth manifold M with coefficients in $\mathbb{R}$, and C_k^{sm} the space of smooth chains. Assume that homology of M are finite-dimensional. **Then cohomology of the complex C_k and cohomology of the complex C_k^{sm} are equal, and the cohomology of the dual complexes are also equal.**

Proof: Fix a complete Riemannian metric on M. Consider the "uniform norm" ν on the space of chains, defined as follows: it is the maximal norm such that for any $f, g : \Delta^k \longrightarrow M$, one has $\nu(\lambda f) \leqslant |\lambda|$ and $\nu(f-g) \leqslant \sup_{x \in \Delta^k} d(f(x), g(x))$. Clearly, the space C_k^{sm} is dense in C_k, hence the cohomology space of C_k^{sm} is dense in cohomology of C_k, which is equal to homology of M. However, cohomology of C_k is finitely-dimensional, hence $H_i(C_k^{sm}) = H_i(M)$. ■

REMARK: Integration over a smooth chain defines a linear map $\Lambda^k(M) \longrightarrow (C_k^{sm})^*$. Stokes' theorem gives $\int_C d\alpha = \int_{\partial C} \alpha$, hence **this map commutes with the differential.**

De Rham theorem

The main result of this lecture: de Rham cohomology are equal to the singular cohomology.

THEOREM: (de Rham theorem)
Let M be a smooth manifold (compact or with a finite polyhedral structure), $H^*_{DR}(M)$ its de Rham cohomology and $H^*_{\mathsf{sing}}(M)$ its singular cohomology. **Then the map $H^*_{DR}(M) \longrightarrow H^*_{\mathsf{sing}}(M)$ constructed above is an isomorphism.**

It will be proven later today.

Mayer-Vietoris theorem for singular homology

CLAIM: Let $M = U \cup V$ be a metrizable space, where U and V are open. Denote by $C^k_{U,V}(M)$ the space of chains generated by simplices $f: \Delta^k \longrightarrow M$ which are contained in V or U. **Then the following sequence of chain complexes is exact:** $0 \longrightarrow C_*(U \cap V) \longrightarrow C_*(U) \oplus C_*(V) \longrightarrow C_*^{U,V}(M) \longrightarrow 0$. ■

CLAIM: The natural embedding $C_*^{U,V}(M) \longrightarrow C_*(M)$ of complexes induces an isomorphism on cohomology of these complexes.

Proof: Fix a metric on M. Any symplex Δ in M can be partitioned onto smaller simplices which lie in U or V. Indeed, let S be a simplex in M, and ε the distance between $S \backslash U$ and $S \backslash V$. These are two non-intersecting compact sets, hence $\varepsilon > 0$. Clearly, any symplex in M of diameter $< \varepsilon$ belongs to U or V or both. Now, if we partition Δ onto smaller simplices of diameter $< \varepsilon$, we obtain a chain $\mathcal{D} \in C_*^{U,V}(M)$. However, $\Delta - \mathcal{D}$ is a boundary. **This construction implies that the map $C_*^{U,V}(M) \longrightarrow C^*(M)$ is surjective on cohomology** of complexes. It is injective on cohomology, because for any boundary $x \in C_*^{U,V}(M)$, $x = \partial(y)$, with $y \in C_*(M)$, y can be partitioned in a similar way, giving $y' \in C_*^{U,V}(M)$ with $d(y') = x'$, where x' is obtained from x by partitionining it onto smaller simplices. ■

Mayer-Vietoris theorem for singular homology and cohomology

COROLLARY: (Mayer-Vietoris exact sequence for homology) Let $M = U \cup V$ be a metrizable space, where U and V are open. **Then there exists a long exact sequence of homology**

$$\begin{aligned}... \longrightarrow H_{i+1}(U) \oplus H_{i+1}(V) \longrightarrow H_{i+1}(U \cup V) \longrightarrow H_i(U \cap V) \longrightarrow \\ \longrightarrow H_i(U) \oplus H_i(V) \longrightarrow H_i(U \cup V) \longrightarrow H_{i-1}(U \cap V) \longrightarrow ...\end{aligned}$$

Proof: We obtain this long exact sequence from the exact sequence of complexes $0 \longrightarrow C_*(U \cap V) \longrightarrow C_*(U) \oplus C_*(V) \longrightarrow C_*^{U,V}(M) \longrightarrow 0$. ■

Dualizing this sequence, we obtain

COROLLARY: (Mayer-Vietoris exact sequence for cohomology) Let $M = U \cup V$ be a metrizable space, where U and V are open. Then **there exists a long exact sequence of singular cohomology with real coefficients.**

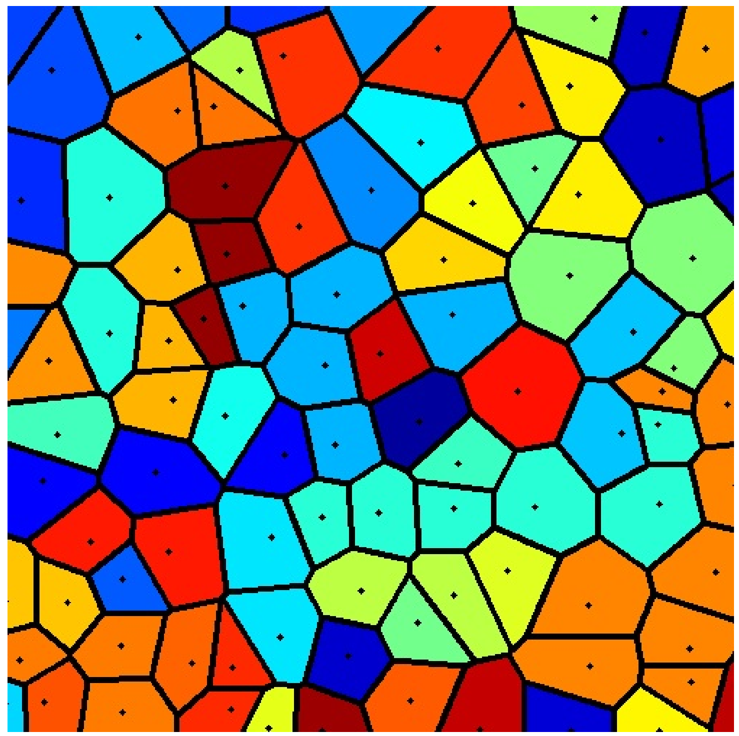
$$\begin{aligned}... \longrightarrow H^{i-1}(U, \mathbb{R}) \oplus H^{i-1}(V, \mathbb{R}) \longrightarrow H^{i-1}(U \cap V, \mathbb{R}) \longrightarrow H^i(U \cup V, \mathbb{R}) \longrightarrow \\ \longrightarrow H^i(U, \mathbb{R}) \oplus H^i(V, \mathbb{R}) \longrightarrow H^i(U \cap V, \mathbb{R}) \longrightarrow H^{i+1}(U \cup V, \mathbb{R}) \longrightarrow ...\end{aligned}$$

■

REMARK: Mayer-Vietoris exact sequence exists for cohomology with any coefficients. The proof is very similar to the one for homology.

Voronoi partitions

DEFINITION: Let M be a metric space, and $S \subset M$ a finite subset. **Voronoi cell** associated with $x_i \in S$ is $\{z \in M \ \ | \ d(z, x_i) \leqslant d(z, x_i) \forall j \neq i\}$. **Voronoi partition** is partition of M onto its Voronoi cells.



Voronoi partition

THEOREM: Let M be a complete Riemannian manifold. **Then, for an appropriate choice of the points x_i, Voronoi cells are polyhedral.**

Convex sets

DEFINITION: Let M be a metric space, and $S \subset M$ a discrete subset. **The Voronoi cell** associated with $x_i \in S$ is $\{z \in M \mid d(z, x_i) \leqslant d(z, x_i) \forall j \neq i\}$. **The Voronoi partition** is partition of M onto its Voronoi cells.

DEFINITION: A subset $S \subset M$ of a Riemannian manifold is called **convex** if for any two points $x, y \subset S$ the minimal geodesic connecting x to y is unique, and it belonds to S.

REMARK: Any convex set S is starlike. Indeed. fix an "origin" $x \in S$. For any $y \in S$, denote by $\gamma_y: [0, d(x,y)] \longrightarrow S$ the minimal geodesic. Then $F_t(y) = \gamma_y(t)$, $t \in [0,1]$ induces a smooth homotopy between identity map and the projection mapping S to the point $\{x\}$.

DEFINITION: Let M be a complete Riemannian manifold. **Injectivity radius** is the supremum of all numbers $\varepsilon > 0$ such that any closed ball of radius ε is convex.

THEOREM: Let M be a compact Riemannian manifold. **Then the injectivity radius of M is always positive.**

Proof: Uses basic differential geometry (left as an exercise). ∎

Convex covers

REMARK: Let M be a compact manifold, and ε less than its injectivity radius. **An ε-net** is a finite subset $S \subset M$ such that M is contained in the union of ε-balls with centers in S. **Then these ε-balls are convex, as well as all their intersections (an intersection of several convex sets is clearly convex).**

DEFINITION: A **convex cover** of a Riemannian manifold is a cover $\{U_i\}$ such that all U_i and all the closures $\overline{U}_i$ are convex.

REMARK: Clearly, **any cover of M by ε-balls is convex,** if ε is less that the injectivity radius of M.

REMARK: If M is a polyhedral manifold, we could take for U_i small neighbourhoods of the polyhedra. Even without the metric, it is clear that the intersections of U_i are starlike for an appropriate choice of neighbourhoods. Indeed, these intersections are neighbourhoods of the corresponding faces of the polyhedra. **This gives a cover with the same properties as a convex cover.**

The proof of de Rham theorem

THEOREM: (de Rham theorem) Let M be a smooth manifold (compact or with a finite polyhedral structure), $H^*_{DR}(M)$ its de Rham cohomology and $H^*_{\text{sing}}(M)$ its singular cohomology. **Then the map $H^*_{DR}(M) \longrightarrow H^*_{\text{sing}}(M)$ constructed above is an isomorphism.**

Proof: Let M be a manifold and $\{U_i\}$ a finite open cover such that all U_i, all the closures $\overline{U}_i$ and all their intersections are starlike (such a cover can be obtained from a polyhedral structure or from a convex cover as above). Poincaré lemma implies that $H^*_{DR}(\overline{K}) \longrightarrow H^*_{\text{sing}}(K)$ is an isomorphism for any K which is starlike, where $\overline{K}$ is the closure of K. Using induction in n, we may assume that $H^*_{DR}(\overline{K}) \longrightarrow H^*_{\text{sing}}(K)$ is an isomorphism for any K which is obtained as a union of $n-1$ or less of U_i. Let us prove it for $K = \bigcup_{i=1}^n U_i$. Using the Mayer-Vietoris exact sequences for de Rham and singular cohomology, applied to $X = U_0$ and $Y = \bigcup_{i=2}^n U_i$, we obtain the following diagram

$$
\begin{array}{ccccccccccc}
... & \longrightarrow & H^{i-1}_{DR}(\overline{X} \cap \overline{Y}) & \longrightarrow & H^{i}_{DR}(\overline{X} \cup \overline{Y}) & \longrightarrow & H^{i}_{DR}(\overline{X}) \oplus H^{i}_{DR}(\overline{X}) & \longrightarrow & H^{i+1}_{DR}(\overline{X} \cap \overline{Y}) & \longrightarrow & ... \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
... & \longrightarrow & H^{i-1}_{\text{sing}}(X \cap Y) & \longrightarrow & H^{i}_{\text{sing}}(X \cup Y) & \longrightarrow & H^{i}_{\text{sing}}(X) \oplus H^{i}_{\text{sing}}(X) & \longrightarrow & H^{i+1}_{\text{sing}}(X \cap Y) & \longrightarrow & ...
\end{array}
$$

By induction assumption, the vertical arrows in this diagram are isomorphisms for all terms except for $H^*_{DR}(\overline{X} \cup \overline{Y}) \longrightarrow H^*_{\text{sing}}(X \cup Y)$. **The 5-lemma implies that they are isomorphisms in all terms.** ■.