# Topologia das Variedades

## Cohomology, lecture 7: Künneth formula

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## Tensor product of complexes

**DEFINITION: Bicomplex** of vector spaces is a sequence of vector spaces $C^{p,q}$ equipped with the differentials $d^{1,0}: C^{p,q} \longrightarrow C^{p+1,q}$ and $d^{0,1}: C^{p,q} \longrightarrow C^{p,q+1}$, such that $d^{0,1} \circ d^{0,1} = d^{1,0} \circ d^{1,0} = 0$ and $d^{0,1} \circ d^{1,0} = d^{1,0} \circ d^{0,1}$.

**EXAMPLE:** Let ... $\xrightarrow{d_C} C^i \xrightarrow{d_C} C^{i+1} \xrightarrow{d_C}$ ... and ... $\xrightarrow{d_D} D^i \xrightarrow{d_D} D^{i+1} \xrightarrow{d_D}$ ... be complexes of vector spaces. Their **tensor product** is the bicomplex $C^p \otimes D^q$ with $d^{1,0} = d_C \otimes \mathrm{Id}_{D^*}$ and $d^{0,1} = \mathrm{Id}_{C^*} \otimes d_D$.

**DEFINITION: Totalization** of a bicomplex $(C^{*,*}, d^{1,0}, d^{0,1})$ is the complex

$$\dots \xrightarrow{d} \bigoplus_{p+q=i} C^{p,q} \xrightarrow{d} \bigoplus_{p+q=i+1} C^{p,q} \xrightarrow{d} \dots$$

where the differential $d$ is defined as $d|_{C^{p,q}} = d^{1,0} + (-1)^p d^{0,1}$.

**REMARK:** To see that $d^2 = 0$, we use $(d^{1,0})^2 = 0$, $(d^{0,1})^2 = 0$ and $d^{1,0}(-1)^p d^{0,1} + (-1)^{p+1} d^{0,1} d^{1,0} = 0$.

## Cohomology and the tensor product

**THEOREM:** Let ... $\xrightarrow{d_C} C^i \xrightarrow{d_C} C^{i+1} \xrightarrow{d_C}$ ... and ... $\xrightarrow{d_D} D^i \xrightarrow{d_D} D^{i+1} \xrightarrow{d_D}$ ... be complexes of vector spaces, and $((C\otimes D)^*, d)$ be the totalization of their tensor product bicomplex. **Then**

$$H^i((C\otimes D)^*) = \bigoplus_{p+q=i} (H^p(C)\otimes H^q(D)). \qquad (***)$$

**Proof. Step 1:** Consider the direct sum decomposition $C^* = C_1^* \oplus C_2^*$. Then $H^i((C\otimes D)^*) = H^i((C_1\otimes D)^*) \oplus H^i((C_2\otimes D)^*)$, and $\bigoplus_{p+q=i}(H^p(C)\otimes H^q(D)) = \bigoplus_{p+q=i}(H^p(C_1)\otimes H^q(D)) \oplus \bigoplus_{p+q=i}(H^p(C_2)\otimes H^q(D))$. Therefore, whenever we represent $C^*$ as a direct sum $\bigoplus_i C_i^*$, **it suffices to prove the isomorphism (***) for each subcomplex $C_i^*\otimes D^*$.**

**Step 2:** Every complex of vector spaces is a direct sum of two types of complexes,

$$... \xrightarrow{d} 0 \xrightarrow{d} V \xrightarrow{d} 0 \xrightarrow{d} ...$$

and

$$... \xrightarrow{d} 0 \xrightarrow{d} V \xrightarrow{d=\operatorname{Id}_V} V \xrightarrow{d} 0 \xrightarrow{d} ...$$

where $V$ is a one-dimensional vector space. It would suffice to prove (***) when $C^*$ and $D^*$ is one of these two types of complexes.

## Cohomology and the tensor product (2)

**THEOREM:** Let $... \xrightarrow{d_C} C^i \xrightarrow{d_C} C^{i+1} \xrightarrow{d_C}$ ... and $... \xrightarrow{d_D} D^i \xrightarrow{d_D} D^{i+1} \xrightarrow{d_D}$ ... be complexes of vector spaces, and $((C \otimes D)^*, d)$ be the totalization of their tensor product bicomplex. **Then**

$$H^i((C \otimes D)^*) = \bigoplus_{p+q=i} (H^p(C) \otimes H^q(D)). \qquad (***)$$

**Step 2:** Every complex of vector spaces is a direct sum of two types of complexes, $... \xrightarrow{d} 0 \xrightarrow{d} V \xrightarrow{d} 0 \xrightarrow{d}$ ... and $... \xrightarrow{d} 0 \xrightarrow{d} V \xrightarrow{d=\operatorname{Id}_V} V \xrightarrow{d} 0 \xrightarrow{d}$ ... where $V$ is a one-dimensional vector space. It would suffice to prove (***) when $C^*$ and $D^*$ is one of these two types of complexes.

**Step 3:** When $C^* = ... \xrightarrow{d} 0 \xrightarrow{d} V \xrightarrow{d} 0 \xrightarrow{d}$ ..., with $V$ in degree 0, we have $(C \otimes D)^* \cong D^*$ and $\bigoplus_{p+q=i}(H^p(C) \otimes H^q(D)) \cong H^i(D)$ hence (***) holds tautologically.

**Step 4:** When both $C^*$ and $D^*$ have form $... \xrightarrow{d} 0 \xrightarrow{d} V \xrightarrow{d=\operatorname{Id}_V} V \xrightarrow{d} 0 \xrightarrow{d}$ ..., the totalization of $C^* \otimes D^*$ takes form

$$... \xrightarrow{d} 0 \xrightarrow{d} V^{\otimes 2} \xrightarrow{d=(\operatorname{Id}_{V^{\otimes 2}}, \operatorname{Id}_{V^{\otimes 2}})} V^{\otimes 2} \oplus V^{\otimes 2} \xrightarrow{d=(\operatorname{Id}_{V^{\otimes 2}}, -\operatorname{Id}_{V^{\otimes 2}})} V^{\otimes 2} \xrightarrow{d} 0 \xrightarrow{d} ...$$

therefore $H^*((C \otimes D)^*) = 0$. Also, $H^*(C) \otimes H^*(D) = 0$, hence (***) also holds. ■

## Tensor product and functions on a product

**THEOREM:** Let $C(M)$ be the space of functions $f: M \longrightarrow \mathbb{R}$, and $C(N)$ the spave of functions $f: N \longrightarrow \mathbb{R}$. Consider the natural map $\Psi: C(M) \otimes C(N) \longrightarrow C(M \times N)$. **Then $\Psi$ is injective.**

**Proof. Step 1: For $N, M$ finite $\Psi$ is an isomorphism.** Indeed, for any $m \in M$ and $n \in N$, the tensor product product $\chi_m \otimes \chi_n$ of atomic functions $\chi_m$ and $\chi_n$ is mapped to $\chi_{(m,n)}$, hence $\Psi$ is surjective, and it is injective because $\dim C(M) \otimes C(N) = |M||N| = \dim C(M \times N)$.

**Step 2:** For any linearly independent set of $k$ functions $f_1, ..., f_k \in C(M)$, consider restriction of $f_1, ..., f_k$ to a finite subset $M_0 \subset M$. If there is a linear relation $\sum_i \lambda_i f_i\big|_{M_0}$ for each finite subset, this linear relation is true on $M$. Therefore, **linearly independent functions remain linearly independent if restricted on a sufficiently big finite subset.**

**Step 3:** Let $\{f_\alpha\}$ be a basis in $C(M)$, $\{g_\alpha\}$ a basis in $C(N)$. Then $\{f_\alpha \otimes g_\alpha\}$ is a basis in $C(M) \otimes C(N)$, indexed by $\alpha \in A$, $\beta \in B$. Any vector $x \in C(M) \otimes C(N)$ takes form $x = \sum_{i \in A_0, j \in B_0} x_{ij} f_i \otimes g_j$, where $A_0 \subset A$, $B_0 \subset B$ are finite subsets. Then $x\big|_{M_0 \times N_0}$ is non-zero for some finite subsets $M_0 \subset M$, $N_0 \subset N$ (Step 2). This implies that $\Psi(x)\big|_{M_0 \times N_0}$ is also non-zero (Step 1). ■

## Stone-Weierstrass theorem

**DEFINITION:** Let $A \subset C^0 M$ be a subspace in the space of continuous functions. We say that $A$ **separates the points** of $M$ if for all distinct points $x, y \in M$, there exists $f \in A$ such that $f(x) \neq f(y)$.

**DEFINITION:** Let $M$ be a topological space, and $\|f\| := \sup_M |f|$ **the sup-norm on functions**. The **$C^0$-topology**, or **uniform topology** on the space $C^0(M)$ of continuous functions is topology defined by the sup-norm.

**THEOREM: (Stone-Weierstrass)**
Let $A \subset C^0 M$ be a subring separating points, and $\overline{A}$ its closure in the $C^0$-topology. **Then $\overline{A} = C^0(M)$.**

**COROLLARY:** For any two manifolds $M$ and $N$, **the ring $C^\infty(M) \otimes C^\infty(N)$ is dense in $C^0(M \times N)$.**

**Proof:** $C^\infty(M) \otimes C^\infty(N)$ is a subring in $C^0(M \times N)$, as shown above, and it is dense because it separates the points. ■

## Differential forms on a product

**CLAIM:** Let $V, W$ be vector spaces. **Then $\Lambda^i(V \oplus W) = \bigoplus_{p+q=i} \Lambda^p(V) \otimes \Lambda^q(W)$.**

**Proof:** The multiplication map $\bigoplus_{p+q=i} \Lambda^p(V) \otimes \Lambda^q(W) \longrightarrow \Lambda^i(V \oplus W)$ takes the monomial basis in $\bigoplus_{p+q=i} \Lambda^p(V) \otimes \Lambda^q(W)$ to the monomial basis in $\Lambda^i(V \oplus W)$.
■

**COROLLARY:** For any two manifolds $M_1$ and $M_2$, consider the projection maps $\pi_i : M_1 \times M_2 \longrightarrow M_i$. Then **the natural multiplication map $\pi_1^* \Lambda^*(M_1) \otimes \pi_2^* \Lambda^*(M_2) \longrightarrow \Lambda^*(M_1 \times M_2)$ is an isomorphism.**

**Proof:** Indeed, $\Lambda^1(M_1 \times M_2) = \pi_1^* \Lambda^1(M_1) \oplus \pi_2^* \Lambda^1(M_2)$, hence

$$\Lambda^i(M_1 \times M_2) = \Lambda^i(\Lambda^1(M_1 \times M_2)) =$$
$$= \Lambda^i\Big(\pi_1^* \Lambda^1(M_1) \oplus \pi_2^* \Lambda^1(M_2)\Big) = \bigoplus_{p+q=i} \pi_1^* \Lambda^p(M_1) \otimes \pi_2^* \Lambda^q(M_2).$$

■

## Differential forms on a product (2)

**COROLLARY:** Consider the natural multiplicative map $\Psi: \bigoplus_{p+q=i} \Lambda^p(M_1) \otimes_{\mathbb{R}} \Lambda^q(M_2) \longrightarrow \Lambda^i(M_1 \times M_2)$ obtained by taking pullbacks and multiplying. **Then $\Psi$ is injective, and its image is dense in $C^0$-topology.**

**Proof:** The image is generated multiplicatively by forms which are lifted from $M_1$ and $M_2$. It would suffice to prove the statement locally in coordinates. Let $V_i$ be the vector spaces generated by $dx_1, ..., dx_n$, $dy_1, ..., dy_m$, where $x_1, ..., x_n$ and $y_1, ..., y_m$ are coordinate functions on $M_i$, defined in a neighbourhood of a point. Then $\Lambda^*(M_1)$ is $C^\infty M_1 \otimes_{\mathbb{R}} \Lambda^*(V_2)$, and $\Lambda^*(M_2)$ is $C^\infty M_2 \otimes_{\mathbb{R}} \Lambda^*(V_2)$. Similarly, $\Lambda^*(M_1 \times M_2) = C^\infty(M_1 \times M_2) \otimes_{\mathbb{R}} \Lambda^*(V_1 \oplus V_2)$. Since $\Lambda^*(M_1) \otimes \Lambda^*(M_2)$ is equal to $C^\infty(M_1) \otimes C^\infty(M_2) \otimes \Lambda^*(V_1 \oplus V_2)$, the second is dense in the first as we have already proven. ■

## Künneth theorem

**THEOREM:** Consider the map $\Psi: \bigoplus_{p+q=i} \Lambda^p(M_1) \otimes_{\mathbb R} \Lambda^q(M_2) \longrightarrow \Lambda^i(M_1 \times M_2)$ obtained by taking pullbacks and multiplying. Define the differential on the totalization of $\Lambda^*(M_1) \otimes_{\mathbb R} \Lambda^*(M_2)$ in the usual way. **Then $\Psi$ commutes with the differential and induces identity on the cohomology.**

*We prove this result later this lecture.*

**COROLLARY: (Künneth theorem)**
For any two manifolds $M_1, M_2$, **there is an isomorphism $H^i(M_1 \times M_2) \cong \bigoplus_{p+q=i} H^p(M_1) \otimes H^q(M_2)$.**

**Proof:** The differential in the totalization of the bicomplex $\Lambda^*(M_1) \otimes_{\mathbb R} \Lambda^*(M_2)$ is compatible with the de Rham differential on $\pi_1^*(\Lambda^*(M_1)) \otimes \pi_2^*(\Lambda^*(M_2)) = \Lambda^*(M_1 \otimes M_2)$. However, cohomology of the latter is $H^*(M_1 \times M_2)$, cohomology of the former is $H^*(M_1) \otimes H^*(M_2)$. Since $\Psi$ induces an isomorphism on cohomology, one has $H^*(M_1 \times M_2) \cong H^*(M_1) \otimes H^*(M_2)$. ■

## Hermann Künneth

Hermann Künneth (1892-1975) was the son of the high school ("Gymnasium", the highest form of high school) teacher Christian Künneth. Beginning with 1910, he studied mathematics at the Universit"at Erlangen and the Ludwig-Maximilians-Universit"at München with a "break" from 1914-1919 where he served in the German army; he was injured twice and was prisoner of war with the British. Künneth was member of the AMV Fridericiana Erlangen, a musically oriented fraternity. His professors in Erlangen were Ernst Sigismund Fischer, Paul Gordan, Max Noether, Richard Baldus and Erhard Schmidt.

1912 he took his first Staatsexamen to become a teacher and 1920 he took his second. In 1920 he became teacher in Bavaria, in particular at high schools ("Gymnasien") in Kronach and Erlangen. He remained in contact with the University in Erlangen, where he got in PhD under the direction of Tietze in 1922 (and was assistant (professor) beginning with 1921). The title of his thesis was "Über die Bettischen Zahlen einer Produktmannigfaltigkeit" - "About the Betti numbers of a product manifold" (where he proved the Künneth formula).

1923 he became assistant (professor) in Berlin; interestingly, he became 1923 also teacher ("Studienrat") in Kronach. As already indicated, the switched to the high school Fridericianum in Erlangen in 1925, where he became Oberstudienrat in 1950 [this would not be a very high position at a high school these days, but I am not sure how it was then]. In 1942 he habilitated in Erlangen and was Privatdozent (a kind of freelancing professor) after that. After he retired in 1957 from his teaching job, he became associate professor in Erlangen. Otto Haupt said about this: "[he] developed an amazing and surprising scientific activity." (at the age of 65)

1964 he got the Bundesverdienstkreuz am Band (the second lowest order of the "Order of Merit of the Federal Republic of Germany"). (See this newspaper article - it says: "His chivalric personality, of clear judgement, emanates human kindness and witty humour.")

## Hermann Künneth (1892-1975)

Hermann Künneth (1892-1975)

## The proof of Künneth theorem

**THEOREM:** Consider the map $\Psi: \bigoplus_{p+q=i} \Lambda^p(M_1) \otimes_{\mathbb{R}} \Lambda^q(M_2) \longrightarrow \Lambda^i(M_1 \times M_2)$ obtained by taking pullbacks and multiplying. Define the differential on the totalization of $\Lambda^*(M_1) \otimes_{\mathbb{R}} \Lambda^*(M_2)$ in the usual way. **Then $\Psi$ commutes with the differential and induces identity on the cohomology.**

**Proof. Step 1:** For any open subset $W \subset M_1 \times M_2$, denote by $\Lambda^*_p(W)$ the algebra of differential forms multiplicatively generated by products of $\pi_1^*(\alpha)$ and $\pi_2^*(\beta)$. As shown above, $\Lambda^*_p(A \times B) = \Lambda^*(A) \otimes_{\mathbb{R}} \Lambda^*(B)$. Denote by $H^*_p(W)$ the cohomology of $(\Lambda^*_p(W), d)$. Since $H^*((C_1 \otimes C_2)^*) = H^*(C_1) \otimes H^*(C_2)$, one has $H^*_p(A \times B) = H^*(A) \otimes H^*(B)$.

**Step 2:** We assume that $M_1, M_2$ are compact or have a finite polyhedral structure. Let $U_1, ..., U_n$ be a cover of $M_1$ and $V_1, ..., V_m$ a cover of $M_2$ with all $U_i$, $V_j$ and all their intersections starlike and closed. As explained in Lecture 6, to build such a cover, one can take small convex neighbourhood of each polyhedra, or the Voronoi cells for an appropriate Riemannian structure. Then $U_i \times V_j$ is a cover of $M_1 \times M_2$ with the same properties. Since $H^*_p(A \times B) = H^*(A) \otimes H^*(B)$, the natural map $\Psi: H^*_p(U_i \times V_j) \longrightarrow H^*(U_i \times V_j)$ is an isomorphism.

## The proof of Künneth theorem (2)

**THEOREM:** Consider the map $\Psi: \bigoplus_{p+q=i} \Lambda^p(M_1) \otimes_{\mathbb R} \Lambda^q(M_2) \longrightarrow \Lambda^i(M_1 \times M_2)$ obtained by taking pullbacks and multiplying. Define the differential on the totalization of $\Lambda^*(M_1) \otimes_{\mathbb R} \Lambda^*(M_2)$ in the usual way. **Then $\Psi$ commutes with the differential and induces identity on the cohomology.**

**Step 3:** The Mayer-Vietoris long exact sequence holds for $H^*_p(\cdot)$-cohomology of closed subsets in $M_1 \times M_2$; the argument is the same as for the usual Mayer-Vietoris sequence. Assume that $\Psi: H^*_p(K) \longrightarrow H^*(K)$ is an isomorphism for all $K \subset M_1 \times M_2$ obtained as union of at most $n-1$ sets obtained by intersections of some of $U_i \times V_j$, hence closed and starlike (indeed, $U \times V \cap U' \times V' = (U \cap U') \times (V \cap V')$).

We prove that $\Psi: H^*_p(K) \longrightarrow H^*(K)$ is an isomorphism for all $n$ using induction by $n$. Let $X = U_i \times V_j$, and $Y$ be the union of $n-1$ such sets. Mayer-Vietoris exact sequence gives

$$\begin{array}{ccccccccc}
... \longrightarrow & H_p^{i-1}(X\cap Y) & \longrightarrow & H^i_p(X\cup Y) & \longrightarrow & H^i_p(X)\oplus H^i_p(X) & \longrightarrow & H_p^{i+1}(X\cap Y) & \longrightarrow ...\\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
... \longrightarrow & H^{i-1}(X\cap Y) & \longrightarrow & H^i(X\cup Y) & \longrightarrow & H^i(X)\oplus H^i(X) & \longrightarrow & H^{i+1}(X\cap Y) & \longrightarrow ...
\end{array}$$

By induction assumption, the vertical arrows in this diagram are isomorphisms for all terms except for $H^*_p(X\cup Y) \longrightarrow H^*(X \cup Y)$. **The 5-lemma implies that they are isomorphisms in all terms.** ■.