Topologia das Variedades

Cohomology, lecture 8: multiplicative structure on cohomology of $\mathbb{C}P^n$

Misha Verbitsky

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Differential graded algebras

DEFINITION: An algebra A is called **graded** if A is represented as $A = \bigoplus A^i$, where $i \in \mathbb{Z}$, and the product satisfies $A^i \cdot A^j \subset A^{i+j}$. Instead of $\bigoplus A^i$ one often writes A^*, where $*$ denotes all indices together. Some of the spaces A^i can be zero, but the ground field is always in A^0, so that it is non-empty.

DEFINITION: Let $A^* = \oplus_{i\in\mathbb{Z}} A^i$ be a graded algebra. For $x \in A^i$, let $\tilde{x} := i$. It is called **graded commutative**, or **supercommutative**, if $ab = (-1)^{\tilde{a}\tilde{b}} ba$.

DEFINITION: **Differential graded algebra**, or **DG-algebra** is a graded commutative algebra $A^* = \oplus_{i\in\mathbb{Z}} A^i$ with a differential $d: \ A^* \longrightarrow A^{*+1}$, $d^2 = 0$, satisfying the Leibnitz identity $d(xy) = (dx)y + (-1)^{\tilde{x}} x(dy)$.

EXAMPLE: De Rham algebra is clearly a DG-algebra.

Cohomology of a differential graded algebras

CLAIM: Let (A^*, d) be a DG-algebra. Then **its cohomology is equipped with a natural graded commutative multiplication.**

Proof: Leibnitz identity implies that a product of closed forms is closed. If $\alpha = d\beta$ and γ is closed, one has $\alpha \wedge \gamma = d(\beta) \wedge \gamma = d(\beta \wedge \gamma)$, hence a product of closed and exact forms is exact. In other words, exact forms are an ideal in the ring of closed forms, and the quotient ring is the ring of cohomology. ■

DEFINITION: **The ring of cohomology of a manifold** is the ring associated with its de Rham algebra.

Symplectic manifolds

DEFINITION: A skew-symmetric 2-form ω on a vector space V is called **non-degenerate** if for any non-zero $x \in V$, there exists $y \in V$ such that $\omega(x,y) \neq 0$.

CLAIM: Let ω be a non-degenerate form on V. **Then there exists a basis $x_1, ..., x_{2n} \in V^*$, such that $\omega = \sum_{i=1}^n x_{2i-1} \wedge x_{2i}$.** ■

DEFINITION: A **symplectic form** on a manifold M is a closed, everywhere non-degenerate 2-form. A manifold with a symplectic form is called **a symplectic manifold**.

REMARK: Symplectic manifold is always oriented. Indeed, at each point where $\omega = \sum_{i=1}^n x_{2i-1} \wedge x_{2i}$, one has $\omega^n = n!\, x_1 \wedge x_2 \wedge ... \wedge x_{2n}$, hence **$\omega^n$ is a nowhere degenerate volume form.**

CLAIM: Let (M,ω) be a compact manifold, $\dim_{\mathbb{R}} M = 2n$. **Then the cohomology class of ω^i, $i = 1, 2, ..., n$ is non-zero.**

Proof: Since ω^n is a volume form, it is non-zero in cohomology. Indeed, otherwise we would have $\omega^n = d\alpha$ and $\int_M \omega^n = \int_M d\alpha = 0$. Therefore, ω^i is also non-zero in cohomology. ■

Homogeneous spaces (reminder)

DEFINITION: A Lie group is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G **acts on a manifold** M if the group action is given by the smooth map $G \times M \longrightarrow M$.

DEFINITION: Let G be a Lie group acting on a manifold M transitively. Then M is called **a homogeneous space**. For any $x \in M$ the subgroup $\mathrm{St}_x(G) = \{g \in G \mid g(x) = x\}$ is called **stabilizer of a point** x, or **isotropy subgroup**.

CLAIM: For any homogeneous manifold M with transitive action of G, **one has $M = G/H$,** where $H = \mathrm{St}_x(G)$ is an isotropy subgroup.

Proof: The natural surjective map $G \longrightarrow M$ putting g to $g(x)$ identifies M with the space of conjugacy classes G/H. ■

REMARK: Let $g(x) = y$. Then $\mathrm{St}_x(G)^g = \mathrm{St}_y(G)$: **all the isotropy groups are conjugate.**

Isotropy representation (reminder)

DEFINITION: Let $M = G/H$ be a homogeneous space, $x \in M$ and $\mathrm{St}_x(G)$ the corresponding stabilizer group. The **isotropy representation** is the natural action of $\mathrm{St}_x(G)$ on T_xM.

DEFINITION: A tensor Φ on a homogeneous manifold $M = G/H$ is called **invariant** if it is mapped to itself by all diffeomorphisms which come from $g \in G$.

REMARK: Let Φ_x be an isotropy invariant tensor on $\mathrm{St}_x(G)$. For any $y \in M$ obtained as $y = g(x)$, consider the tensor Φ_y on T_yM obtained as $\Phi_y := g(\Phi)$. The choice of g is not unique, however, for another $g' \in G$ which satisfies $g'(x) = y$, we have $g = g'h$ where $h \in \mathrm{St}_x(G)$. Since Φ is h-invariant, **the tensor Φ_y is independent from the choice of g.**

We proved

Theorem 1: **G-invariant tensors on $M = G/H$ are in bijective correspondence with isotropy invariant tensors on T_xM,** for any $x \in M$. ■

Representations acting transitively on a sphere

THEOREM: Let G be a group acting on a vector space V with scalar product g. Suppose that G acts transitively on a unit sphere $\{x \in V \ \ | \ \ g(x) = 1\}$. **Then a G-invariant bilinear symmetric form is unique up to a constant multiplier.**

Proof. Step 1: Since G preserves the sphere, which is a level set of the quadratic form g, g is G-invariant.

Step 2: For any G-invariant quadratic form g', the function $x \longrightarrow \frac{g'(x)}{g(x)}$ is constant on spheres and invariant under homothety, hence it is constant. ■

Complex projective space

DEFINITION: Let $V = \mathbb{C}^n$ be a complex vector space equipped with a Hermitian form h, and $U(n)$ the group of complex endomorphisms of V preserving h. This group is called **the complex isometry group**.

DEFINITION: Complex projective space $\mathbb{C}P^n$ is the space of 1-dimensional subspaces (lines) in $\mathbb{C}^{n+1}$.

REMARK: Since $U(n+1)$ acts on lines transitively, **$\mathbb{C}P^n$ is a homogeneous space, $\mathbb{C}P^n = \frac{U(n+1)}{U(1)\times U(n)}$,** where $U(1) \times U(n)$ is a stabilizer of a line in $\mathbb{C}^{n+1}$.

EXAMPLE: $\mathbb{C}P^1$ is S^2.

Fubini-Study form

EXAMPLE: Consider the natural action of the unitary group $U(n+1)$ on $\mathbb{C}P^n$. The stabilizer of a point is $U(n) \times U(1)$.

THEOREM: There exists an $U(n+1)$-invariant Riemann metric on $\mathbb{C}P^n$. Moreover, **such a metric is unique up to a constant multiplier.**

Proof. Step 1: To construct a $U(n+1)$-invariant Riemann form on $\mathbb{C}P^n$, we take a $U(n)$-invariant form on $T_x\mathbb{C}P^n$ and apply Theorem 1. To show that a $U(n)$-invariant form on $T_x\mathbb{C}P^n$ exists, take any Riemannian form and average it with $U(n)$-action.

Step 2: Uniqueness follows because the isotropy group acts transitively on a sphere. ■

REMARK: This Riemannian structure is called **the Fubini-Study metric**.

Fubini-Study form

DEFINITION: Let $v \in \mathbb{C}P^n$ be a point, which we consider as 1-dimensional subspace in $\mathbb{C}^{n+1}$. Then $T_v\mathbb{C}P^n = \operatorname{Hom}_{\mathbb{C}}(v, v^\perp)$, hence it is a complex vector space. Denote by $I: T_v\mathbb{C}P^n \longrightarrow T_v\mathbb{C}P^n$ the corresponding **complex structure operator**, $I^2 = -\operatorname{Id}$.

REMARK: Since $I \in U(1) \subset U(n) \times U(1)$ (stabilizer of a point), the Fubini-Study metric is I-invariant: $g(Ix, Iy) = g(x, y)$. This gives

$$g(x, Iy) = g(Ix, I^2y) = -g(Ix, y) = -g(y, Ix).$$

In other words, the form $\omega(x, y) := g(x, Iy)$ is anti-symmetric.

DEFINITION: Let g be a Fubini-Study metric on $\mathbb{C}P^n$. The 2-form $\omega(x, y) := g(x, Iy)$ is called **the Fubini-Study form**. **It is clearly non-degenerate.**

CLAIM: The Fubini-Study form on $\mathbb{C}P^n$ is symplectic.

Proof: Let ω be a Fubini-Study form. Then $d\omega$ is an isotropy-invariant 3-form on $T_x\mathbb{C}P^n$. However, the isotropy group contains $-\operatorname{Id}$, **hence all isotropy-invariant odd tensors vanish.** ■

Projective manifolds

DEFINITION: A projective manifold is a submanifold of $\mathbb{C}P^n$ obtained as the set of solutions of a system homogeneous polynomial equations $P_1(z_1, ..., z_{n+1}) = P_2(z_1, ..., z_{n+1}) = ... = P_k(z_1, ..., z_{n+1}) = 0$.

REMARK: Let $Z \subset \mathbb{C}P^n$ be a projective manifold. Then $T_v Z$ is the set of vectors $x \in T_v\mathbb{C}P^n$ such that the differentials of P_i (in appropriate affine coordinates) vanish on x. Therefore, $I(T_vZ) = T_vZ$.

COROLLARY: Let $Z \subset \mathbb{C}P^n$ be a projective manifold, and $g|_Z$ be restriction of the Fubini-Study form. **Then $\omega_Z(x, y) := g(x, Iy)$ defines a symplectic form on Z.**

Proof: The form ω_Z is non-degenerate because $g|_Z$ is positive definite, and closed because restriction of a closed form is closed (this is because restriction is pullback, and pullback commutes with the de Rham differential). ■

Homogeneous and affine coordinates on $\mathbb{C}P^n$

DEFINITION: We identify $\mathbb{C}P^1$ with the set of $n+1$-tuples $x_0 : x_1 : ... : x_n$ defined up to equivalence $x_0 : x_1 : ... : x_n \sim \lambda x_0 : \lambda x_1 : ... : \lambda x_n$, for each $\lambda \in \mathbb{C}^*$. This representation is called **homogeneous coordimates**. **Affine coordinates** in the chart $x_k \neq 0$ are are $\frac{x_0}{x_k} : \frac{x_1}{x_k} : ... : 1 : ... : \frac{x_n}{x_k}$; the space $\mathbb{C}P^n$ is a union of $n+1$ affine charts identified with $\mathbb{C}^n$, with the complement to each chart identified with $\mathbb{C}P^{n-1}$.

DEFINITION: Let $\pi : \mathbb{C}^{n+1} \backslash 0 \longrightarrow \mathbb{C}P^n$ be the natural projection. **The Hopf map** $H : S^{2n+1} \longrightarrow \mathbb{C}P^n$ is the restriction of this map to a unit sphere.

CLAIM: Identify the complement of affine chart $1 : x_1 : ... : x_n$ with $\mathbb{C}P^{n-1}$, using the homogeneous coordinates $0 : x_1 : ... : x_n$. Let $(x_0, x_1, ..., x_n) \in S^{2n+1} \subset \mathbb{C}^{n+1}$ be a point on the unit sphere, distinct from $(1, 0, 0, ..., 0)$. Then $\lim_{t \to 0} (tx_0 : x_1 : ... : x_n) = H(x_1 : ... : x_n)$. ■

COROLLARY: **$\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by gluing a cell homeomorphic to a unit ball $B \subset \mathbb{C}P^n$,** with $x \in \partial B = S^{2n+1}$ glued to $H(x)$. ■

Cellular homology

DEFINITION: A k-dimensional CW-complex is defined inductively as follows. A 1-dimensional CW-complex is a graph. To define a k-dimensional CW-complex X_k, let X_{k-1} be a $k-1$-dimensional CW-complex and let $\{B_i\}$ be a collection of k-dimensional closed balls. For each of B_i fix a continuous map $\delta_i: \partial B_i \longrightarrow X_{k-1}$. Then X_k is obtained as a quotient space of $X_{k-1} \coprod_i B_i$ by a relation $v \sim \delta_i(v)$ for each $v \in \partial B_i$; in other words, X_k is obtained by gluing the cells $\{B_i\}$ to X_{k-1} using the maps δ_i.

DEFINITION: Let $Z_{k-1} \subset X_{k-1}$ be a $k-1$-dimensional cell of a CW-complex X, and $Z_k \subset X_k$ its k-dimensional cell. Consider the map $\Psi_{Z_{k-1}}: X_{k-1} \longrightarrow S^{k-1}$ obtained by gluing the boundary of Z_{k-1}, X_{k-2} and the rest of the $k-1$-dimensional cells to a point. Then the composition of $\delta_{Z_k}: S^{k-1} = \partial Z_k \longrightarrow X_{k-1}$ and $\Psi_{Z_{k-1}}: X_{k-1} \longrightarrow S^{k-1}$ is a map from spere to a sphere. Denote by $\delta(Z_k, Z_{k-1})$ its degree. The **cell complex** of X is the complex (C^*, d), where C^k is a free abelian group generated by k-cells, and $d(Z_k) = \sum_{R_i} \delta(Z_k, R_i)$, where R_i runs from all $k-1$-cells such that $\delta(Z_k, R_i)$ is non-zero (there are finitely many).

Cellular homology of $\mathbb{C}P^n$

THEOREM: Cohomology of the cell complex of X are equal to its singular cohomology.

Proof: It was proven in lectures by M. B. ■

REMARK: In the cellular decomposition of $\mathbb{C}P^n$ there are $n+1$ cells, with one in each even dimension $i = 0, 2, 4, ..., 2n$. Therefore, $H_i(\mathbb{C}P^n) = \mathbb{Z}$ when i is even, $0 \leqslant i \leqslant 2n$, and $H_i(\mathbb{C}P^n) = 0$ when i is odd.

COROLLARY: The cohomology algebra of $\mathbb{C}P^n$ is the algebra of truncated polynomials: $\mathbb{R}[\omega]/[\omega]^{n+1}$, where ω is the Fubini-Study form and $[\omega]$ its cohomology class.

Proof: Since cohomology are dual to homology, one has $H^i(\mathbb{C}P^n, \mathbb{R}) = \mathbb{R}$ when i is even, $0 \leqslant i \leqslant 2n$, and $H^i(\mathbb{C}P^n) = 0$ when i is odd. Since ω^i is never cohomologous to 0, it generates $H^{2i}(\mathbb{C}P^n, \mathbb{R})$. ■