Topologia das Variedades

Cohomology, lecture 9: Poincaré duality

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Cohomology with compact support

DEFINITION: Let ω be a differential form on M **Support** Supp(ω) of ω is the closure of the set of all points where $\omega \neq 0$. We say that ω **has compact support** if its support is compact.

REMARK: Clearly, **a differential of a form with compact support has compact support.**

DEFINITION: **Cohomology with compact support** of a manifold M is cohomology of the complex $(\Lambda_c^*(M), d)$ of differential forms with compact support.

REMARK: Let $U \subset M$ be an open subset. **Then $\Lambda_c^*(U) \subset \Lambda_c^*(M)$.** Indeed, any two non-intersecting closed subsets of a manifold can be separated by non-intersecting open subsets. Applying this to Supp(α) and $M\backslash U$, we obtain an open set $V \supset M\backslash U$ such that $\alpha|_{V\cap U} = 0$. Extend α to V by setting $\alpha = 0$ on the whole V, and we obtain a smooth extension of α to M.

REMARK: This defines a map $H_c^*(U) \longrightarrow H_c^*(M)$. **Cohomology is contravariant functor** (the functor which inverts the arrows), **cohomology with compact support is covariant functor**.

Long exact sequence (reminder)

DEFINITION: A complex is a sequence of vector spaces and homomorphisms $... \xrightarrow{d} C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \xrightarrow{d} ...$ satisfying $d^2 = 0$. **Homomorphism** $(C^*, d) \longrightarrow (C_1^*, d)$ of complexes is a sequence of homomorphism $C^i \longrightarrow C_1^i$ commuting with the differentials.

DEFINITION: An element $c \in C^i$ is called **closed** if $c \in \ker d$ and **exact** if $c \in \operatorname{im} d$. **Cohomology** of a complex is a quotient $\frac{\ker d}{\operatorname{im} d}$. One denotes the i-th group of cohomology of a complex by $H^i(C^*)$

REMARK: A homomorphism of complexes induces a natural homomorphism of cohomology groups.

DEFINITION: Short exact sequence of complexes $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ is a sequence of morphisms of complexes $A^* \xrightarrow{x} B^* \xrightarrow{y} C^*$ such that $x: A^i \longrightarrow B^i$ is injective, $y: B^i \longrightarrow C^i$ is surjective (for all i), and $\ker y = \operatorname{im} x$.

THEOREM: Let $0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$ be an exact sequence of complexes. Then there exists a **long exact sequence of cohomology**

$$... \longrightarrow H^{i-1}(C^*) \longrightarrow H^i(A^*) \longrightarrow H^i(B^*) \longrightarrow H^i(C^*) \longrightarrow H^{i+1}(A^*) \longrightarrow ...$$

Differential forms on closed subsets (reminder)

DEFINITION: Let M be a manifold, and $Z \subset M$ a closed subset. Let α, β be two differential forms defined in open sets U_α and U_β containing Z. We say that α and β are equivalent if $\alpha = \beta$ on $U_\alpha \cap U_\beta$. The space of equivalence classes is denoted $\Lambda^*(Z)$ and called **the space of differential forms on Z** or **the space of germs of differentials forms in Z**.

REMARK: If Z is a manifold with boundary, $\Lambda^*(Z)$ is the space of differential forms on Z, by definition of differential forms on manifolds with boundary.

EXERCISE: Using partition of unity, **prove that the natural restriction map $\Lambda^*(M) \longrightarrow \Lambda^*(Z)$ is surjective for any closed $Z \subset M$.**

REMARK: The de Rham differential is well defined on $\Lambda^*(Z)$, allowing us to **define the de Rham cohomology of Z as usual.** Poincaré lemma holds in this situation, too.

CLAIM: Let $X \subset \mathbb{R}^n$ be a closed starlike subset. **Then $H^i(X) = 0$ for all $i > 0$.**

Proof: Same as for open X. ■

Cohomology of a closed subset

REMARK: Let $Z \subset M$ be a closed subset of a manifold. **The notation $\Lambda_c^*(Z)$ is slightly misleading,** because the forms in this space are defined in a neighbourhood of Z in M. To avoid confusion, we denote the cohomology of this complex by $H_Z^*(M)$.

CLAIM: Let $Z \subset M$ be a smooth subvariety. Assume that there exists a neighbourhood $U \supset Z$ and a family of smooth maps $\varphi_t : U \longrightarrow U$ smoothly depending on $t \in [0,1]$ such that φ_0 is identity and $\varphi_1 : U \longrightarrow Z$ is identity on Z. **Then $H_Z^*(M) = H^*(Z) = H^*(U)$.**

Proof: The maps φ_t induce the same map on cohomology of U and on $H_Z^*(M)$ as follows from the homotopy invariance of de Rham cohomology (same argument as used in the proof of Poincaré lemma). Then $\varphi_1 : U \longrightarrow U$ induces an identity map on $H_Z^*(M)$ and on $H^*(U)$. Therefore, the composition $U \xrightarrow{\varphi_1} Z \hookrightarrow U$ induces identity on cohomology, both $H^*(U)$ and $H_Z^*(U)$. The composition $Z \hookrightarrow U \xrightarrow{\varphi_1} Z$ is the identity map. We obtain that $H^*(U) = H_Z^*(U) = H^*(Z)$. ■

Excision for cohomology with compact support

PROPOSITION: Let $Z \subset M$ be a closed subset of a manifold. **Then the following sequence of complexes is exact.**

$$0 \longrightarrow \Lambda_c^*(M \backslash Z) \longrightarrow \Lambda_c^*(M) \xrightarrow{R} \Lambda_c^*(Z) \longrightarrow 0.$$

Proof: Surjectivity in the last term and injectivity in the first is clear. Now, let $R(\alpha) = 0$, where α is a form with compact support. Then $\alpha = 0$ in a neighbourhood of Z. This means that $\operatorname{Supp}(\alpha) \subset M \backslash Z$. ■

COROLLARY: (Excision exact sequence)
Let $Z \subset M$ be a compact subset of a manifold. **There is a long exact sequence**

$$... \longrightarrow H_Z^i(M) \longrightarrow H_c^{i+1}(M \backslash Z) \longrightarrow H_c^{i+1}(M) \longrightarrow H_Z^{i+1}(M) \longrightarrow H_c^{i+2}(M \backslash Z) \longrightarrow ...$$

■

Cohomology of $\mathbb{R}^n$ with compact support

COROLLARY: (Excision exact sequence)
Let $Z \subset M$ be a compact subset of a manifold. **There is a long exact sequence**

$$... \longrightarrow H^i_Z(M) \longrightarrow H^{i+1}_c(M\backslash Z) \longrightarrow H^{i+1}_c(M) \longrightarrow H^{i+1}_Z(M) \longrightarrow H^{i+2}_c(M\backslash Z) \longrightarrow ...$$

■

Let us apply this to $M = S^n$ and Z a point. Using the previous claim, we can assume that $H^*_Z(M) = H^*(pt)$, where $H^*(pt)$ denotes cohomology of a point. Then

$$... \longrightarrow H^i(pt) \longrightarrow H^{i+1}_c(\mathbb{R}^n) \longrightarrow H^{i+1}_c(S^n) \longrightarrow H^{i+1}(pt) \longrightarrow H^{i+2}_c(\mathbb{R}^n) \longrightarrow ...$$

Positive-dimensional cohomology of a point vanish, and the only non-zero cohomology of S^n are in dimension n. Therefore, all positive-dimensional terms in this exact sequence vanish, except in dimension n, where we get $H^{n-1}(pt) = \longrightarrow H^n_c(\mathbb{R}^n) \longrightarrow H^n(S^n) = \mathbb{R} \longrightarrow H^n(pt) = 0$, hence $H^n_c(\mathbb{R}^n) = \mathbb{R}$.

COROLLARY: $H^i_c(\mathbb{R}^n) = \mathbb{R}$ for $i = n$ and vanishes otherwise.

Proof: Since the closed 0-forms are constant functions, and they don't have compact support on non-compact manifold, $H^0_c(\mathbb{R}^n) = 0$. The rest of cohomology vanish except $H^n_c(\mathbb{R}^n) = \mathbb{R}$ as shown above. ■

Poincaré lemma with compact support

LEMMA: Let $M \subset \mathbb{R}^n$ be an open starlike set. Then **Then $H^i_c(M) = \mathbb{R}$ for $i = n$ and vanishes otherwise.**

Proof. Step 1: Consider the map $\Lambda^n_c(M) \longrightarrow \mathbb{R}$ mapping α to $\int_M \alpha$. Clearly, it is non-zero on positively oriented n-forms with compact support and zero on $d(\Lambda^{n-1}_c(M))$ by Stokes' theorem. To prove the Poincaré lemma with compact support, **it remains to show that all the n-forms α with $\int_M \alpha = 0$ and all closed i-forms with $i < n$ are exact.**

Step 2: Denote by Γ^t the homothety map $z \longrightarrow tz$, where $0 < t \leqslant 1$. Then Γ^t maps forms with compact support to forms with compact support; denote this map by $\Gamma^t_* : \Lambda^i_c(M) \longrightarrow \Lambda^i_c(M)$. Clearly $\frac{\Gamma^t_*}{dt} = -\operatorname{Lie}_{\vec{r}}$, where $\vec{r}$ is a radial vector field. Since Lie_x acts by zero on cohomology, **Γ^t_* acts on $H^*_c(M)$ as identity.**

Step 3: Let $\alpha \in \Lambda^i_c(M)$ be an n-form with $\int_M \alpha = 0$ or a closed i-form with $i < n$. Since α is cohomologous to 0 in $H^*_c(\mathbb{R}^n)$, one has $\alpha = d\beta$, where β is a form on $\mathbb{R}^n$ with compact support. Choose t in such a way that support of $\Gamma^t_*\beta$ belongs to M. **Then $\Gamma^t_*\alpha$ is cohomologous to 0 in $H^*_c(M)$.** However, the cohomology class of α is equal to the cohomology class of $\Gamma^t_*\alpha$ (Step 2).

■

Mayer-Vietoris with compact support (1)

CLAIM: Consider open subsets U, V of a manifold M, Then **the natural embedding maps define an exact sequence of complexes:**

$$0 \longrightarrow \Lambda_c^*(U \cup V) \xrightarrow{\varphi} \Lambda_c^*(U) \oplus \Lambda_c^*(V) \xrightarrow{\psi} \Lambda_c^*(U \cup V) \longrightarrow 0. \quad (*)$$

Here φ is the natural embedding on both components, and ψ is the natural embedding on the first component and its opposite on the second component.

Proof. Step 1: The map φ is clearly injective. Also, the sequence is exact in the middle term, because for any $\lambda \in \Lambda_c^*(U)$, $\mu \in \Lambda_c^*(V)$ such that $\lambda - \mu = 0$ in $\Lambda_c^*(U \cup V)$, one has $\operatorname{Supp}(\lambda) = \operatorname{Supp}(\mu) \subset U \cup V$.

Step 2: It remains to prove that ψ is surjective. Let $Z = \operatorname{Supp}(\alpha)$ be a compact subset of a form $\alpha \in \Lambda_c^*(U \cup V)$, and let $Z_U := Z\backslash V$, $Z_V := Z\backslash U$. These are non-intersecting compact subsets, hence there exists a covering $\{U_i\}$ such that none of U_i intersects both Z_V and Z_U. Let ψ_i be the corresponding partition of unity, let Ψ_U be the sum of all φ_i intersecting Z_U and Ψ_V be the sum of the rest of φ_i. Then $\Psi_U\big|_{Z_V} = 0$ and $\Psi_V\big|_{Z_U} = 0$, hence $\Psi_U\alpha \in \Lambda_c^*(U)$ and $\Psi_V\alpha \in \Lambda_c^*(V)$. We have decomposed α to a sum of two forms which lie in $\Lambda_c^*(U)$ and $\Lambda_c^*(V)$. ■

Mayer-Vietoris with compact support (2)

CLAIM: Consider open subsets U, V of a manifold M, Then **the natural embedding maps define an exact sequence of complexes:**

$$0 \longrightarrow \Lambda_c^*(U \cap V) \xrightarrow{\varphi} \Lambda_c^*(U) \oplus \Lambda_c^*(V) \xrightarrow{\psi} \Lambda_c^*(U \cup V) \longrightarrow 0. \quad (*)$$

Here φ is the natural embedding on both components, and ψ is the natural embedding on the first component and its opposite on the second component.

COROLLARY: (Mayer-Vietoris with compact support)
Let $U, V \subset M$ be open subsets. **Then there is a long exact sequence associated with (*):**

$$... \longrightarrow H_c^{i-1}(U) \oplus H_c^{i-1}(U) \longrightarrow H_c^{i-1}(U \cup V) \longrightarrow H_c^i(U \cap V) \longrightarrow$$
$$\longrightarrow H_c^i(U) \oplus H_c^i(V) \longrightarrow H_c^i(U \cup V) \longrightarrow H_c^{i+1}(U \cap V) \longrightarrow ...$$

■

Poincaré duality

REMARK: Let M be a manifold. Consider the multiplication map $\Lambda^*(M) \otimes \Lambda^*_c(M) \longrightarrow \Lambda^*_c(M)$. As we have already seen, a product of closed forms is closed, and a product of a closed and exact forms is exact.

THEOREM: (Poincaré duality theorem)
Let M be an n-dimensional oriented connected manifold. Then the integration map $\alpha \longrightarrow \int_M \alpha$ gives an isomorphism $H^n_c(M) = \mathbb{R}$. Moreover, **the multiplication $H^i_c(M) \times H^{n-i}(M) \longrightarrow H^n_c(M) = \mathbb{R}$ defines a non-degenerate pairing between $H^i_c(M)$ and $H^{n-i}(M)$.**

It will be proven later today

REMARK: On a compact manifold, $H^i_c(M) = H^i(M)$, and then the **Poincaré pairing** $\alpha, \beta \longrightarrow \int_M \alpha \wedge \beta$ defines a non-degenerate bilinear form **(the Poincaré form)** on cohomology.

REMARK: Note that **1-dimensionality of $H^n_c(M)$ is itself a consequence of Poincare duality,** because $H^n_c(M)$ is dual to $H^0(M) = \mathbb{R}$.

Proof of Poincaré duality

THEOREM: (Poincaré duality theorem)
Let M be an n-dimensional oriented connected manifold. Then the integration map $\alpha \longrightarrow \int_M \alpha$ gives an isomorphism $H^n_c(M) = \mathbb{R}$. Moreover, **the multiplication $H^i_c(M) \times H^{n-i}(M) \longrightarrow H^n_c(M) = \mathbb{R}$ defines a non-degenerate pairing between $H^i_c(M)$ and $H^{n-i}(M)$.**

Proof: Let M be a manifold and $\{U_i\}$ a finite open cover such that all U_i, all the closures $\overline{U}_i$ and all their intersections are starlike (such a cover can be obtained from a polyhedral structure or from a convex cover as in Lecture 6). Poincaré lemma implies that the Poincare pairing map $H^*(\overline{K}) \longrightarrow H^{n-*}_c(\overline{K})^*$ is an isomorphism for any K which is starlike, where $\overline{K}$ is the closure of K. Using induction in n, we may assume that $H^*(\overline{K}) \longrightarrow H^{n-*}_c(\overline{K})^*$ is an isomorphism for any K which is obtained as a union of $n-1$ or less of U_i. Let us prove it for $K = \bigcup_{i=1}^n U_i$. Mayer-Vietoris exact sequences, applied to $X = U_0$ and $Y = \bigcup_{i=2}^n U_i$, give the following diagram.

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H^{i-1}(\overline{X} \cap \overline{Y}) & \longrightarrow & H^i(\overline{X} \cup \overline{Y}) & \longrightarrow & H^i(\overline{X}) \oplus H^i(\overline{X}) & \longrightarrow & H^{i+1}(\overline{X} \cap \overline{Y}) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_c^{n-i+1}(X \cap Y)^* & \longrightarrow & H_c^{n-i}(X \cup Y)^* & \longrightarrow & H_c^{n-i}(X)^* \oplus H_c^{n-i}(Y)^* & \longrightarrow & H_c^{n-i-1}(X \cap Y)^* & \longrightarrow & \dots
\end{array}$$

By induction assumption, the vertical arrows in this diagram are isomorphisms for all terms except possibly for $H^*(\overline{X} \cup \overline{Y}) \longrightarrow H^{n-*}_c(X \cup Y)^*$. **The 5-lemma implies that they are isomorphisms in all terms.** ■.