Topologia das Variedades

Cohomology, lecture 10: Hopf theorem

Misha Verbitsky

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Bialgegras

DEFINITION: Let $A^\bullet$, $B^\bullet$ be graded commutative algebras The **tensor product algebra** is $A^\bullet \otimes B^\bullet$ with the product $a \otimes b \cdot a' \otimes b' = (-1)^{\tilde{b}\tilde{a}'} aa' \otimes bb'$.

REMARK: By Künneth formula, $H^\bullet(X \times Y)$ is isomorphic to $H^\bullet(X) \otimes H^\bullet(Y)$ as an algebra.

DEFINITION: Let $A^\bullet$ be a graded commutative algebra over a field k. We say that $A^\bullet$ is a **bialgebra** if it is equipped with a homomorphism of algebras $A \xrightarrow{\Delta} A \otimes A$, called **comultiplication** which is **coassociative**, that is, satisfies

$$\Delta \circ \Delta \otimes \operatorname{Id}_A = \Delta \circ \operatorname{Id}_A \otimes \Delta: \ A \longrightarrow A \otimes_k A \otimes_k A.$$

Counit of a bialgebra is an algebra homomorphism $A \xrightarrow{\varepsilon} k$ which satisfies $\Delta \circ (\varepsilon \otimes \operatorname{Id}_A) = \Delta \circ (\operatorname{Id}_A \otimes \varepsilon) = \operatorname{Id}_A$ In the sequel, we shall tacitly assume that all bialgebras have counit.

REMARK: **Coassociative comultiplication means that the dual space $(A^\bullet)^*$ is equipped with an algebra structure.** Compatibility of comultiplication with the multiplication in $A^\bullet$ means that **this algebra structure on $(A^\bullet)^*$ is given by a morphism of A-modules.**

Examples of bialgebras

EXAMPLE: Let N be a set equipped with an associative operation $N \times N \xrightarrow{m} N$ with unit e (such a structure is called **the structure of a monoid**, or **semigroup with unit** Then **the ring of k-valued functions $C(N)$ is a bialgebra**, with comultiplication morphism given by $m^*: C(N) \longrightarrow C(N \times N) = C(N) \otimes_k C(N)$, and counit $\varepsilon(v) = v(e)$.

REMARK: The notion of a bialgebra is an abstraction of this observation: heuristically speaking, **bialgebras are alrebras of functions on monoids**.

EXAMPLE: Let N be a topological space equipped with a continuous map $N \times N \xrightarrow{m} N$ inducing the structure of a monoid. Consider the comultiplication on the cohomology algebra $H^\bullet(N)$, given by $m^*: H^\bullet(N) \longrightarrow H^\bullet(N \times N) = H^\bullet(N) \otimes_k H^\bullet(N)$. **Then $H^\bullet(N)$ is a bialgebra**. Indeed, coassociativity of m^* follows from associativity of N, and counit is given by the pullback to $H^\bullet(e) = H^0(e) = k$.

H-spaces

DEFINITION: An H-space is a topological space M equipped with a continuous map $M \times M \xrightarrow{\mu} M$ ("the multiplication map") and an element $e \in M$ ("the unit") which satisfy "semigroup conditions up to homotopy", namely the following.

* **Homotopy associativity:** the maps $\mu \times \operatorname{Id} \circ mu: \ M \times M \times M \longrightarrow M$ and $\operatorname{Id} \times \mu \circ \mu: \ M \times M \times M \longrightarrow M$ are homotopic.

* **Homotopy unit:** the map $\mu: \ M \times \{e\} \longrightarrow M$ is homotopic to identity.

EXAMPLE: Clearly, **any toplogical group is an H-space.**

CLAIM: Let M be an H-space. **Then the cohomology algebra $H^\bullet(M)$ is a bialgebra.** ■

Loop spaces as H-spaces

EXAMPLE: Let $\Omega(M, x)$ be the space of loops, that is, paths $\gamma: [0,1] \longrightarrow M$ starting and ending in x. We can multiply loops by mapping a pair $\gamma_1, \gamma_2 : [0,1] \longrightarrow M$ to a loop $\gamma_1\gamma_2$ $[0,1] \longrightarrow M$ equal to $\gamma_1(2t)$ on $[0,1/2]$ and to $\gamma_2(2t-1)$ on $[1/2,1]$. The homotopy unit is the constant loop. This defines **the structure of an H-group on the loop space**.

REMARK: The topology on the loop space $\Omega(M, x)$ can be defined, for example, **by assuming that M is a metric space, and consider the uniform topology on the maps $\gamma: [0,1] \longrightarrow M$.** In more generality, we take **the compact-open topology**, with the base sets consisting of all loops which map a given compact $K \subset [0,1]$ to an open set $U \subset M$.

Bialgebras of finite type

Let $V^\bullet$ be a graded vector space. Denote by $\operatorname{Sym}_{gr}(V^\bullet)$ the tensor product $\operatorname{Sym}^*(V^{\text{even}}) \otimes \Lambda^*(V^{\text{odd}})$ with a natural grading. On $\operatorname{Sym}^*(V^{\text{even}}) \otimes \Lambda^*(V^{\text{odd}})$ one has a natural structure of an algebra.

DEFINITION: Free commutative algebra is a polynomial algebra. **Free graded commutative algebra** is $Sym_{gr}(V^\bullet)$, where $V^\bullet$ is a graded vector space.

DEFINITION: A graded algebra of finite type is an algebra graded by $i \geqslant 0$, with all graded components finitely-dimensional.

THEOREM: (Hopf theorem) Let $A^\bullet$ be a graded bialgebra of finite type over a field k of characteristic 0. **Then $A^\bullet$ is a free graded commutative k-algebra**.

REMARK: This allows one to compute the multiplicative structure **on all Lie groups and on all loop spaces of finite CW-spaces.**

REMARK: For any Lie group $H^*(G)$ is finite-dimensional, hence **$H^*(G)$ is isomorphic to a Grassmann algebra.** In particular, $\dim H^*(G) = 2^n$.

Heinz Hopf (1894-1971)

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Primitive elements in a bialgebra

DEFINITION: An element x of a bialgebra is called **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$.

REMARK: First, the Hopf theorem is proven for all bialgebras, generated (multiplicatively) by the primitive elements, and then we prove that finite type bialgebras are generated by primitive elements.

DEFINITION: Let A be a Hopf algebra, and $P \subset A$ the space of primitive elements. Consider the natural homomorphism $\operatorname{Sym}_{gr}(P) \xrightarrow{\psi} A$. We say that A **is free up to degree k** if $\bigoplus^{i \leqslant k} \operatorname{Sym}^i_{gr}(P) \xrightarrow{\psi} A$ is an embedding.

REMARK: The following lemma **immediately implies Hopf theorem for all bialgebras generated by primitive elements.**

LEMMA: Let $A^\bullet$ be a bialgebra which is free up to degree k. **Then $A^\bullet$ is free up to degree $k+1$.**

Hopf theorem for bialgebras generated by primitive elements

LEMMA: Let $A^\bullet$ be a bialgebra which is free up to degree k. **Then $A^\bullet$ is free up to degree $k+1$.**

Proof. Step 1: Let $\{x_i\}$ be a basis in the space P of primitive elements. Consider a polynomial relation of degree $k+1$, say, $Q(x_1,...,x_n)=0$, and represent it as a polynomial of x_1 with coefficients which are polynomials of $x_2,...,x_n$: $Q=Q_m x_1^m+Q_{m-1}x_1^{m-1}+...+Q_0$. Clearly, $\Delta(Q)=Q\otimes 1+1\otimes Q+R$, where $R\in\mathfrak{A}:=\left(\bigoplus^{i\leqslant k}\operatorname{Sym}^i_{gr}(P)\right)\otimes\left(\bigoplus^{i\leqslant k}\operatorname{Sym}^i_{gr}(P)\right)$.

Step 2: Since $\psi:\ \bigoplus^{i\leqslant k}\operatorname{Sym}^i_{gr}(P)\xrightarrow{\psi}A$ is an embedding, and elements of $\mathfrak{A}$ belong to $\operatorname{im}\psi\otimes\operatorname{im}\psi$, each element of $\mathfrak{A}$ can be uniquely represented as a sum of monomials $\lambda\otimes\mu$, where λ,μ are degree $\leqslant k$ monomials on x_i. Denote by $\Pi:\ \mathfrak{A}\longrightarrow x_1\otimes\left(\bigoplus^{i\leqslant k}\operatorname{Sym}^i_{gr}(P)\right)$ the projection to the sum of all monomials of form $x_1\otimes\mu$. Since $\Delta(x_i)=x_i\otimes 1+1\otimes x_i$, one has $\Delta(x_1^m)=(x_1\otimes 1+1\otimes x_1)^m$, **giving $\Pi(\Delta(x_1^m))=mx_1\otimes x_1^{m-1}$.**

Step 3: Let $\Pi(R):=x_1\otimes R_0$. Since $Q=0$ in A, its component R_0 is also equal to 0. Then Step 2 gives $0=x_1\otimes R_0=x_1\otimes\sum_{i=1}^m mx_1^{m-1}Q_m$ where Q_i are polynomials defined in Step 1. Then all $Q_i=0$. ■

Algebras with filtration

REMARK: Step 3 of the proof of previous lemma uses $\operatorname{char} k = 0$. **Hopf theorem is false for $\operatorname{char} k > 0$.**

DEFINITION: Filtration on an algebra A is a sequence of subspaces $A_0 \supset A_1 \supset A_2 \supset ...$ such that $A_i \cdot A_i \subset A_{i+j}$

EXAMPLE: Let $I \subset A$ be an ideal. **the I-adic filtration** is the filtration by the degrees of the ideal I: $A \supset I \supset I^2 \supset I^3 \supset ....$

DEFINITION: Let $A_0 \supset A_1 \supset A_2 \supset ...$ be a filtered algebra. **The associated graded algebra** is $A_{gr} := \bigoplus_i A_i/A_{i+1}$.

LEMMA: Let $A \supset I \supset I^2 \supset I^3 \supset ...$ be an adic filtration, and $A_{gr} := \bigoplus_i I^i/I^{i+1}$ the associated graded algebra. **Then A_{gr} is generated by its first and second graded components $A/I \oplus I/I^2$.**

Proof: Indeed, I^k/I^{k+1} is generated by products of k elements in (I/I^2). ∎

Proof of Hopf theorem

DEFINITION: Augmentation ideal in a bialgebra is the kernel of the counit homomorphism $\varepsilon : A \longrightarrow k$.

REMARK: The counit condition gives $x = [\varepsilon \otimes \mathrm{Id}_A](\Delta(x))$ and $x = [\mathrm{Id}_A \otimes \varepsilon](\Delta(x))$, hence

$$\Delta(x) = 1 \otimes x + x \otimes 1 \mod (Z \otimes Z)$$

THEOREM: (Hopf theorem) Let A be a graded bialgebra of finite type over a field k of characteristic 0. **Then A is a free graded commutative k-algebra**.

Proof. Step 1: Consider the filtration of A by the degrees of the augmentation ideal A, and let $A_{gr} := \bigoplus_i Z^i/Z^{i+1}$ be the associated graded algebra. **Since $\Delta(x) = 1 \otimes x + x \otimes 1 \mod (Z \otimes Z)$, one has $\Delta(Z^p) \subset \bigoplus_{i+j=p} Z^i \otimes Z^j$.**

Step 2: This implies that all bialgebra operations on A are compatible with the Z-adic filtration. **Therefore, A_{gr} is also a Hopf algebra**.

Step 3: The algebra A_{gr} is multiplicatively generated by Z^1/Z^2. Since $\Delta(x) = 1 \otimes x + x \otimes 1 \mod (Z \otimes Z)$, all elements of Z^1/Z^2 are primitive in A_{gr}. Therefore, the algebra A_{gr} is generated by primitive elements. This implies that A_{gr} **is a free algebra generated by its space of primitive elements.**

Proof of Hopf theorem (2)

Step 4: Let x_i be a basis in the space of primitive elements of A_{gr}, and let $\tilde{x}_i$ be a representative of each of $x_i \in Z^k/Z^{k+1}$ in Z_k, of the same parity as x_i. **Since there is no non-trivial relations between x_i, there are no non-trivial relations between $\tilde{x}_i$.** It remains to show that $\tilde{x}_i$ generate A.

Step 5: Return to the grading originally given on A. Since ε is compatible with grading, the ideal Z is a direct sum of its graded components, and the algebra A_{gr} is equipped with a grading induced from A. Dimensions of the graded components A^p and A^p_{gr} of A and A_{gr} are equal, because any filtered space is isomorphic as a vector space to its associated graded space. Let $\{y_i\}$ be a set of monomials of $x_i \in A_{br}$ giving a basis in the graded component A^p_{gr}, and $\{\tilde{y}_i\}$ the corresponding monomials in A^p. **Since $\{y_i\}$ are linearly independent, the monomials $\{\tilde{y}_i\}$ are linealry independent, and since $\dim A^p = \dim A^p_{gr}$, these monomials generate A^p.** We have shown that $A^\bullet$ is freely generated by the vectors $\{\tilde{y}_i\}$. ■