Topologia das Variedades

Cohomology, lecture 11: fundamental class

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Cohomology with compact support (reminder)

DEFINITION: Let ω be a differential form on M **Support** Supp(ω) of ω is the closure of the set of all points where $\omega \neq 0$. We say that ω **has compact support** if its support is compact.

REMARK: Clearly, **a differential of a form with compact support has compact support.**

DEFINITION: **Cohomology with compact support** of a manifold M is cohomology of the complex $(\Lambda_c^*(M), d)$ of differential forms with compact support.

THEOREM: **(Poincaré duality theorem)**
Let M be an n-dimensional oriented connected manifold. Then the integration map $\alpha \longrightarrow \int_M \alpha$ gives an isomorphism $H_c^n(M) = \mathbb{R}$. Moreover, **the multiplication $H_c^i(M) \times H^{n-i}(M) \longrightarrow H_c^n(M) = \mathbb{R}$ defines a non-degenerate pairing between $H_c^i(M)$ and $H^{n-i}(M)$.**

Fundamental class

DEFINITION: Let M be an oriented n-manifold, $X \subset M$ a d-dimensional compact oriented submanifold. Consider a functional $H^d(M) \xrightarrow{\int_X} \mathbb{R}$ mapping a form $\alpha \in H^d(M)$ to $\int_X \alpha$. Since $H^d(M)^* = H_c^{n-d}(M)$, the linear functional $\int_X$ is given by the integration with a cohomology class $[X]$:

$$\int_X \alpha = \int_M [X] \wedge \alpha$$

Then $[X]$ is called **the fundamental class of M**.

DEFINITION: A de Rham cohomology class is called **integer** if it is an image of an integer singular class under the natural map $H^*(M, \mathbb{Z}) \longrightarrow H^*(M, \mathbb{R})$.

CLAIM: Fundamental class $[X]$ is always integer.

Proof: Partition $[X]$ into a union of simplices Δ_i. Then $\int_\alpha X = \sum_i \int_{\Delta_i} \alpha$, that is, it is given by a pairing with an integer homology class. ■

Intersection index

DEFINITION: Let $X, Y \subset M$ be two closed submanifolds of an n-manifold with $\dim X + \dim Y = n + d$. We say that X and Y **intersect transversally** if for $d < 0$ they do not intersect and for $d \geqslant 0$ and for each intersection point $m \in X \cap Y$, there exists a neighbourhood $U \ni m$ and a diffeomorphism to an open ball $U \longrightarrow B$ which maps X and Y to linear subspaces, intersecting in a d-dimensional subspace.

DEFINITION: Let $X, Y \subset M$ be two closed oriented submanifolds of an oriented n-manifold with $\dim X = a$, $\dim Y = b$ and $a + b = n$. Assume that X and Y intersect transversally. Then $X \cap Y$ is a discrete set, and for any $m \in X \cap Y$, one has $T_m M = T_m X \oplus T_m Y$. Using orientation on X and Y, we obtain the volume forms $\omega_X \in \Lambda^a T_m^* X$ and $\omega_Y \in \Lambda^b T_m^* Y$, defined up to a positive constant. Then $\omega_X \wedge \omega_Y \in \Lambda^n T_m^* M$ defines an orientation in $T_m M$. We say that **X and Y intersect positively in m** if this form defines the standard orientation on M, and **intersect negatively** if it defines the opposite orientation.

REMARK: If X and Y are odd-dimensional, $\omega_X \wedge \omega_Y = -\omega_Y \wedge \omega_X$, hence **for odd dimensions the intersection index depends on order,** otherwise it is independent.

Intersection form

The main result of today's lecture.

THEOREM: Let $X, Y \subset M$ be two compact oriented submanifolds of an oriented n-manifold with $\dim X + \dim Y = n$, and $[X], [Y]$ their fundamental classes. Assume that they intersect transversally in $x_1, ..., x_n$, with intersection indexes $\sigma_1, ..., \sigma_n \in \{\pm 1\}$. **Then $\int_M [X] \wedge [Y] = \sum_i \sigma_i$.**

In other words, **Poincaré pairing is equal to the intersection pairing**.

It would be proven later today.

A tubular neighbourhood

DEFINITION: Let $X \subset M$ be a smooth submanifold, and $TM|_X$ the tangent bundle to M restricted to X. The **normal bundle** NX is defined as a quotient $TM|_X/TX$. **A tubular neighbourhood** of X in M is a neighbourhood $U \supset X$ which is diffeomorphic to a neighbourhood of X in the total space of NX which is convex in each fiber N_xX.

CLAIM: Let $X \subset M$ be a compact submanifold. **Then a tubular neighbourhood always exists.**

Proof: Introduce a Riemannian metric on M. Then NX can be identified with an orthogonal complement to TX in $TM|_X$. Consider the exponential map $\exp: \operatorname{Tot} NX \longrightarrow M$ mapping $v \in NX$ to $\gamma_v(1)$, where $\gamma: [0,1] \longrightarrow M$ is a geodesic tangent to v. It is well defined when v is sufficiently small, and its differential in $v = 0$ is bijective, because it is bijective when M is a vector space with a flat Riemannian metric and $X \subset M$ is an affine subspace. **Then $\exp: \operatorname{Tot} NX \longrightarrow M$ is a diffeomorphism to its image in a neighbourhood of $X \subset \operatorname{Tot} NX$** by the inverse function theorem. ∎

Realizing the fundamental class by a form

REMARK: Let $X \subset M$ be a submanifold and $U \supset X$ a tubular neighbourhood. Realizing U as a fiberwise convex subset in $\operatorname{Tot}(NX)$ and using the homothety map, we obtain that **U is homotopy equivalent to $X \subset \operatorname{Tot}(NX)$.**

Poincaré duality immediately implies the following claim.

CLAIM: Let $X \subset M$ be a submanifold and $U \supset X$ a tubular neighbourhood. **Then the space $H^d(X)$ is dual to $H_c^{n-d}(U)$.**

REMARK: The n-th cohomology of a compact oriented n-manifold X is 1-dimensional, by Poincare duality. By definition, **the group $H^n(X)$ is generated by its own funfamental class $[X]$.**

PROPOSITION: Let $X \subset M$ be a compact d-dimensional submanifold, with X and M oriented and X compact Consider a tubular neighbourhood of $U \supset X$. **Then the fundamental class of X can be represented by a compactly supported closed form $\tau_X \in \Lambda_c^{n-d}(U)$.**

Proof: Let $\alpha \in H^d(M)$ be any closed form. Then $\int_X \alpha$ depends only on restriction of α to U. Using Poincare duality in $H^*(U)$, we find a class in $H_c^{n-d}(U)$ represented by a form $\tau_X \in \Lambda_c^{n-d}(U)$ such that $\int_X \alpha = \int_U \tau_X \wedge \alpha$. Then $\tau_X = [X]$. ■

Approximating the δ-function

DEFINITION: Let $x \in M$ be a point. **δ_x-function** is a functional on the space $C^0(M)$ of bounded functions such that $\delta_x(f) = f(x)$.

DEFINITION: Consider a sequence α_i of top forms on M with $\int_M \alpha_i = 1$. We say that α_i **approximates the δ-function** if $\lim_i \int \alpha_i f = f(x)$.

PROPOSITION: Let $\pi : W \longrightarrow M$ be a smooth fibration, β a form with compact support of dimension $\dim W - \dim M$, and α_i approximates the delta-function δ_x Then $\lim_i \int \pi^* \alpha_i \wedge \beta = \int_{\pi^{-1}(x)} \beta$.

Proof: Follows from the Fubini theorem. ■

Intersection form and Poincaré duality

THEOREM: Let $X, Y \subset M$ be two compact oriented submanifolds of an oriented n-manifold with $\dim X + \dim Y = n$, and $[X], [Y]$ their fundamental classes. Assume that they intersect transversally in $x_1, ..., x_n$, with intersection indexes $\sigma_1, ..., \sigma_n \in \{\pm 1\}$. **Then $\int_M [X] \wedge [Y] = \sum_i \sigma_i$.**

Proof. Step 1: Let U be a tubular neighbourhood of X in M, and $\tau_X \in \Lambda_c^{n-d}(U)$ the form representing the fundamental class. We need only to show that $\int_Y \tau_X = \sum_i \sigma_i$.

Step 2: Consider the projection $\pi : U \longrightarrow X$ obtained from the identification between U and an open subset in $\operatorname{Tot}(NX)$. Applying an appropriate diffeomorphism, we may assume that $Y \cap U$ is a union of fibers of π. Then $\int_M [X] \wedge [Y] = \int_M \tau_X \wedge [Y] = \int_Y \tau_X = \sum_i \int_{\pi^{-1}(x_i)} \tau_X$. Therefore, the statement of the theorem would follow if we prove that $\int_{\pi^{-1}(x)} \tau_X = 1$ for any $x \in X$.

Step 3: Assume that X is d-dimensional. For any $\alpha \in \Lambda^d(X)$, one has $\int_U \alpha \wedge \tau_X = \int_X \alpha$. Consider a sequence of forms $\alpha_i \in \Lambda^d(X)$ approximating the δ-function; then $\lim_i \int_X \alpha_i \wedge \beta = \int_{\pi^{-1}(x)} \beta$. This gives $\int_{\pi^{-1}(x)} \tau_X = \lim_i \alpha_i \wedge \tau_x = \int_X \alpha_i = 1$ ■