Topologia das Variedades: June 29 exam

The final mark is given as follows: A+ for score of 6, A for 4, B for 3, C for 2, and modified (increased or decreased) depending on results of the oral examination.

The first student (of two) is offered all odd problems and all problems with "2 points" mark, the second student even problems and with "2 points" mark.

Exercise 1.1 (2 points). Let M be a compact simply connected manifold, and $Z \subset M$ a closed smooth hypersurface. Prove that the complement $M \backslash Z$ is disconnected, or find a counterexample

Exercise 1.2 (2 points). Prove that $\mathbb{R}^5$ is not homeomorphic to $\mathbb{R}^4$.

Exercise 1.3. Prove that a smooth map $\phi: S^4 \longrightarrow S^5$ cannot be surjective.

Exercise 1.4. Let $D \subset \mathbb{R}^2$ be an open disk. Prove that a smooth map $\phi: S^4 \longrightarrow D$ cannot be surjective.

Exercise 1.5. Let G be a topological group. Prove that its fundamental group is commutative.

Exercise 1.6. Find a compact manifold with fundamental group $\mathbb{Z}/n\mathbb{Z}$, for any given number $n \in \mathbb{Z}$.

Exercise 1.7. Let G be a finite group which freely acts on a manifold M, and $M_1 = M/G$. Prove that the pullback map of de Rham cohomology $H^*(M_1) \longrightarrow H^*(M)$ is injective.

Exercise 1.8. Let $A \in GL(n, \mathbb{R})$ be a linear automorphism with all eigenvalues $|a_i| > 1$, and $H := \mathbb{R}^n \backslash 0 / \langle A \rangle$ be a quotient of $\mathbb{R}^n \backslash 0$ by the group generated by A. Prove that H is diffeomorphic to $S^1 \times S^{n-1}$.

Exercise 1.9. Let $V = \mathbb{R}^{2n}$, and $\alpha \in \Lambda^n V$ be a non-zero element of the corresponding Grassmann algebra. Prove that there exists $\beta \in \Lambda^n V$ such that $\alpha \wedge \beta \neq 0$.

Exercise 1.10. Let $\omega \in \Lambda^2 V$ be a non-degenerate anti-symmetric 2-form on a vector space. Prove that the multiplication by ω defines an injective map $\Lambda^1(V) \longrightarrow \Lambda^3(V)$.

Exercise 1.11. Calculate $\pi_3(S^3 \times RP^4)$.

Exercise 1.12. Show that a closed surface whose fundamental group G is infinite is $K(G, 1)$.