Topologia das Variedades: July 03 exam

Each student gets a list of 6 problems chosen randomly. Regular ones are worth 1 point, with "2 points" problems where indicated. Please write the solutions, and be prepared to explain them orally using your notes.

The final mark is given as follows: A+ for score of 5, A for 4, B for 3, C for 2, and modified (increased or decreased) depending on results of the oral examination (with additional questions if required).

1 Homology and cohomology

Exercise 1.1. Let K be a Klein bottle, that is, a manifold obtained by gluing two ends of a cylinder with opposite orientation. Find $H^*(K, \mathbb{Z})$.

Exercise 1.2 (2 points). Let M be a compact, smooth n-manifold, and $f : M \longrightarrow S^n$ a smooth map of degree 0. Prove that f is homotopic to a projection to a point.

Exercise 1.3. Prove that any homology class in $H_2(S^2 \times S^2, \mathbb{Z})$ can be represented by a smooth submanifold.

Exercise 1.4. Prove that any homology class in $H_2(\mathbb{C}P^2, \mathbb{Z})$ can be represented by a smooth submanifold.

Exercise 1.5. Find an automorphism A of an n-torus T^n such that any smooth curve fixed by A is homologous to 0.

Exercise 1.6. Let X be a connected CW-space, and SX its suspension. Prove that $H_i(X) = H_{i+1}(SX)$ for all $i > 0$.

Exercise 1.7. Compute cohomology groups of the space $S^1 \vee S^1 \vee S^1$ and of its universal cover.

Exercise 1.8. Compute cohomology of $S^1 \times (S^1 \vee S^2)$.

Exercise 1.9. Show that the map $S^1 \times S^1 \longrightarrow (S^1 \times S^1 / S^1 \vee S^1) \cong S^2$ is not homotopic to a constant map.

2 De Rham cohomology

Exercise 2.1. Prove that any continuous map from S^{2n} to $\mathbb{C}P^n$ has degree 0.

Exercise 2.2. Prove that any continuous map from $S^2 \times S^2$ to $\mathbb{C}P^2$ has degree 0.

Exercise 2.3. Prove that any continuous map from $\mathbb{C}P^n$ to a compact Lie group of dimension $2n$ has degree 0.

Exercise 2.4. Prove that any continuous map from a compact, simply connected Lie group of dimension $2n$ to $\mathbb{C}P^n$ has degree 0.

Exercise 2.5. Prove that any continuous map from a sphere S^{2n} to a compact, simply connected Lie group of dimension $2n$ has degree 0.

Exercise 2.6 (2 points). Let $X, Y \subset \mathbb{C}P^2$ be complex algebraic curves, that is, submanifolds given by a homogeneous polynomial equations. Assume that X and Y are non-empty; prove that $X \cap Y \neq \emptyset$.

Exercise 2.7 (2 points). Let $\omega \in \Lambda^2(V)$, $V = \mathbb{R}^{2n}$ be a non-degenerate 2-form. Prove that multiplication by ω induces an isomorphism $\Lambda^{n-1}(V) \xrightarrow{\wedge\omega} \Lambda^{n+1}(V)$.

Exercise 2.8. Let $\omega \in \Lambda^2(V)$, $V = \mathbb{R}^{2n}$ be a non-degenerate 2-form. Prove that multiplication by ω^{n-1} induces an isomorphism $\Lambda^1(V) \xrightarrow{\wedge\omega^{n-1}} \Lambda^{2n-1}(V)$.

Exercise 2.9. Let A be an automorphism of $\mathbb{C}P^3$ preserving the orientation. Prove that A acts as identity on its de Rham cohomology.

3 Homotopy groups

Exercise 3.1. Calculate $\pi_3(S^3 \times \mathbb{R}P^4)$.

Exercise 3.2 (2 points). Calculate $\pi_3(S^3 \vee \mathbb{R}P^4)$.

Exercise 3.3 (2 points). Prove that $\pi_1(\mathbb{R}P^2 \vee \mathbb{R}P^2)$ is non-commutative.

Exercise 3.4 (2 points). Let L_1, L_2 be non-intersecting lines in $\mathbb{R}^3$, and $M := \mathbb{R}^3 \backslash (L_1 \cap L_2)$ the complement. Prove that $\pi_2(M) = 0$.

Exercise 3.5. Let $M \subset \mathbb{R}^3$ be the set of all points with at least two of three coordinates irrational. Prove that it is connected.

Exercise 3.6. Let $M = \mathbb{R}^2 \backslash \{0, 1\}$. Probe that $\pi_2(M) = 0$.

Exercise 3.7. Find a compact 3-dimensional manifold M such that $\pi_3(M) = 0$.

Exercise 3.8 (2 points). Construct a compact manifold M such that $\pi_1(M)$ is a free group $\mathbb{F}_2$.

Exercise 3.9. Find all n such that there exists a surjective map $S^n \longrightarrow S^n$ of degree 0.