# VHS, home assignment: Hodge decomposition on a torus

**Rules:** This is a home assignment for the next week. We shall discuss the solutions next Tuesday.

**Exercise 1.1.** Let $G$ be a connected Lie group acting on a compact Riemannian manifold $M$ by isometries. Using the uniqueness of the harmonic representative in any cohomology class, prove that all harmonic forms on $M$ are $G$-invariant.

**Exercise 1.2.** Let $T = \mathbb{R}^n/\mathbb{Z}^n$ be a compact torus, equipped with a constant metric. We consider $T$ as a group equipped with the natural free, transitive action on itself. Using the uniqueness of the harmonic representative in any cohomology class, prove that all harmonic forms on $T$ are $T$-invariant. Prove that all $T$-invariant forms are harmonic.

**Exercise 1.3.** Let $V$ be a real vector space. Prove that the Hodge structures of weight 1 on $V$ are in bijective correspondence with linear operators $I: V \longrightarrow V$ such that $I^2 = -\operatorname{Id}$.

**Exercise 1.4.** Let $V$ be a real vector space, $\dim V = 2n$ and $\mathfrak{W}$ the set of all Hodge structures of weight 1 on $V$. Identify $\mathfrak{W}$ with an open subset in the Grassmannian $\operatorname{Gr}_{\mathbb{C}}(n, 2n)$. Prove that $\dim_{\mathbb{C}} \mathfrak{W} = n^2$.

**Exercise 1.5.** Let $T = W/\mathbb{Z}^{2n}$ be a compact complex torus, $W = \mathbb{C}^n$ and $V = (\mathbb{R}^{2n}, I)$ be the same space considered as real space, with $I$ acting as $\sqrt{-1}\operatorname{Id}$. Prove that $H^1(T, \mathbb{R}) = V^*$, and this identification is compatible with the Hodge decomposition on $V^*$.

**Definition 1.1.** Let $Z \subset M$ be a $d$-dimensional submanifold in an $n$-dimensional compact manifold. The **fundamental class** of $Z$ is defined as a class $[Z] \in H^{n-d}(M)$ which satisfies $\int_M \eta \wedge [Z] = \int_Z \eta$ for all $\eta \in H^d(M)$.

**Exercise 1.6.** Prove that the fundamental class is defined uniquely by this expression. Prove that $[Z] \in H^{n-d}(M, \mathbb{Z})$.

**Exercise 1.7.** Construct a complex manifold $M$ and a non-trivial complex submanifold $Z \subset M$ such that $[Z] = 0$.

**Exercise 1.8.** Let $Z \subset M$ be a $p$-dimensional complex submanifold in an $n$-dimensional compact Kähler manifold. Prove that $[Z]$ is an integer class of type $(n-p, n-p)$. Prove that $[Z] \neq 0$.

**Exercise 1.9.** Let $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ be a Hodge structure of weight 1 on $V = \mathbb{R}^4$. Prove that $V$ is uniquely determined by the line $\Lambda^{2,0}(V) \subset \Lambda^2_{\mathbb{C}}(V_{\mathbb{C}})$.

**Exercise 1.10.** Prove that the space all Hodge structures of weight 1 on $V = \mathbb{R}^4$ is identified with a subvariety $Q \subset \mathbb{P}\Lambda^2 V_{\mathbb{C}}$ where $Q$ is defined by the equations $\Omega \wedge \Omega = 0$, $\Omega \wedge \bar{\Omega} > 0$.

**Hint.** Use the previous exercise, and set $\Omega$ as a generator of $\Lambda^{2,0}(V)$.

**Exercise 1.11.** Using this identification, prove that a vector $\alpha \in \Lambda^2 V$ is of type (1,1) if and only if $\alpha \wedge \Omega = 0$ and $\alpha \wedge \bar{\Omega} = 0$.

**Exercise 1.12.** Let $\mathfrak{W}_0$ be the space of all Hodge structures of weight 1 on $V = \mathbb{R}^4$ such that a given non-zero class $\alpha \in \Lambda^2 V$ has type (2,2), and $\mathfrak{W}$ be the space of all Hodge structures of weight 1 on $V$. Prove that $\dim_{\mathbb{R}} \mathfrak{W}_0 = \dim_{\mathbb{R}} \mathfrak{W} - 2$.

**Exercise 1.13.** Prove that there exists a complex torus of complex dimension 2 which does not contain complex curves.

**Hint.** Use the previous exercise and Exercise 1.8.