VHS, home assignment: Isotrivial elliptic fibrations

Rules: This is a home assignment for the next week. We shall discuss the solutions next Tuesday.

2.1 Elliptic fibrations

Definition 2.1. **Elliptic curve** is a compact complex torus of complex dimension 1 with marked point.

Remark 2.1. The distinction between "elliptic curves" and "complex tori of dimension 1" becomes important if we speak about fibrations: fibrations which admit a holomorphic section and which don't admit sections have different nature. Nevertheless, any fibration with fiber a 1-dimensional compact complex torus is called **an elliptic fibration**.

Definition 2.2. A holomorphic fibration is called **isotrivial** if all its fibers are isomorphic.

Exercise 2.1. Let T be a compact complex torus, and G the group of complex automorphisms which are homotopy equivalent to identity. Prove that G is the group of parallel translations of T.

Exercise 2.2. Let E be a complex torus of dimension 1. Consider a sheaf $\mathcal{A}$ on X which takes an open set $U \subset X$ to the group of holomorphic maps $U \mapsto \operatorname{Aut}(E)$. Let $\pi: M \longrightarrow X$ be a smooth, proper holomorphic fibration with fiber E.

a. Prove that the isomorphism classes of such fibrations are in bijective correspondence with $H^1(\mathcal{A})$ (usually this bijective correspondence is expressed as "fibrations are classified by $H^1(\mathcal{A})$").

b. Suppose that $\pi: M \longrightarrow X$ a smooth, proper holomorphic fibration with fiber E. Assume, moreover, that the monodromy on the cohomology of the fibers is trivial. Prove that such fibrations are classified by $H^1(\mathcal{A}_0)$, where $\mathcal{A}_0$ is the sheaf of maps $U \mapsto \operatorname{Aut}_0(E)$, where $\operatorname{Aut}_0(E)$ is the group of parallel translations of E.

c. Construct a sheaf exact sequence $0 \longrightarrow \mathbb{Z}^2_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}_0 \longrightarrow 0$. Prove that this gives an exact sequence $H^1(X, \mathbb{Z}^2) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(\mathcal{A}_0) \xrightarrow{\tau} H^2(X, \mathbb{Z}^2)$. Prove that the map $H^1(X, \mathbb{Z}^2) \longrightarrow H^1(X, \mathcal{O}_X)$ is injective when $\pi_1(X) = 0$.

d. Find an example of a holomorphic fibration such that $\operatorname{im} \tau$ is not torsion.

Remark 2.2. When $\dim_{\mathbb{C}} M = 2$, A. Blanchard (1956) has shown that $\operatorname{im} \tau$ is non-torsion if and only if M is non-Kähler. It is a good exercise for a student who mastered Kähler geometry.

Exercise 2.3. Let $\pi: M \longrightarrow X$ be a smooth, proper holomorphic map, with fiber a complex torus of complex dimension 1. Assume that π is isotrivial and admits a section, and $\pi_1(X) = 0$. Prove that $M = X \times E$, where E is an elliptic curve.

Exercise 2.4. Construct an isotrivial, non-trivial fibration with fiber and base isomorphic to a complex torus of complex dimension 1, admitting a section.

Exercise 2.5. Construct an isotrivial, non-trivial fibration $\pi: M \longrightarrow X$ with fiber a complex torus of complex dimension 1, and base which satisfies $\pi_1(X) = 0$, not admitting a section.

2.2 Algebraic dimension of complex tori

Definition 2.3. A projective complex torus with a marked point is called **an abelian variety**. Note that *an abelian variety is always a group.*

Remark 2.3. We shall freely use the following theorem, due to Kodaira. Let M be a compact complex manifold, and ω a Kähler form which has integer cohomology class. Then $\omega = c_1(L)$, where L is a holomorphic line bundle on M. Moreover, L is ample, and M is projective.

Definition 2.4. Let M be a complex manifold, and ω a real (1,1)-form. We say that ω is **semi-positive** if $\omega(x, Ix) \geqslant 0$ for any $x \in TM$.

Definition 2.5. A differential form on a torus is called **constant** it it is invariant under parallel translations.

Exercise 2.6. Let M be a complex torus, and ω a closed semipositive (1,1)-form. Prove that ω is homologous to a constant, semipositive (1,1)-form.

Exercise 2.7. Let M be a compact complex torus, and $\omega \in \Lambda^2 M$ a closed semipositive (1,1)-form with integer cohomology class. Prove that there exists a smooth surjective holomorphic map $\pi : M \longrightarrow N$ to another complex torus and a Kähler form ω_0 on N such that $\pi^*\omega_0$ is cohomologous to ω.

Definition 2.6. Algebraic dimension $a(M)$ of a compact torus M is the maximal dimension of a projective variety X such that there exists a surjective holomorphic map $\pi : M \longrightarrow X$.

Remark 2.4. Generally, algebraic dimension of M is the transcendence dimension of the field $k(M)$ of meromorphic functions on M. For M a complex torus, this definition is equivalent to the one given above. A student who mastered Kähler geometry can prove that for a torus these two definitions are equivalent (good exercise).

Exercise 2.8. Let M be a compact torus of dimension 1. Prove that $a(M) = 1$.

Exercise 2.9. Let M be a compact torus such that $\dim_{\mathbb{C}} M = a(M)$. Prove that M is projective.

Exercise 2.10. Let M be a compact torus, and $2d$ the maximal rank of a constant semipositive form with integer class in $H^2(M)$ and of type (1,1) on M. Prove that $d = a(M)$.

Exercise 2.11. Let M be a compact Kähler manifold.

a. Let $H_1(M, \mathbb{Z})_{tf}$ denote the torsion-free part of the cohomology. We consider it as a lattice in $H^1(M, \mathbb{R})$. Consider a natural map $H_1(M, \mathbb{Z})_{tf} \mapsto H^{1,0}(M)^*$ taking a class $\gamma \in H_1(M, \mathbb{Z})_{tf}$ to the map $\int_\gamma \in H^{1,0}(M)^*$. Prove that it is injective.

b. Denote by $\operatorname{Alb}(M)$ the quotient $\frac{H^{1,0}(M)^*}{H_1(M,\mathbb{Z})_{tf}}$. Prove that it is compact.

c. Construct a holomorphic map $a : M \mapsto \operatorname{Alb}(M)$ such that $a^*(H^1(\operatorname{Alb}(M))) = H^1(M)$. Prove that such map is unique up to a translation of $\operatorname{Alb}(M)$. The map a is called **the Albanese map.**

d. Let M be a compact torus equipped with a complex structure of Kähler type. Prove that the Albanese map $a : M \mapsto \operatorname{Alb}(M)$ is biholomorphic.

Exercise 2.12. Using the previous exercize, prove that the Teichmüller space Teich of complex structures of Kähler type on a compact torus is $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$. Prove that it is equal to the period space of the Hodge structures of weight 1 on $H^1(M, \mathbb{R})$. Prove that $\dim_{\mathbb{R}} \operatorname{Teich} = 2n^2$.

Exercise 2.13. Let $\eta \in H^2(M, \mathbb{R})$ be a non-zero cohomology class on a torus, $\dim_{\mathbb{R}} M = 4$, and $\operatorname{Teich}_\eta \subset \operatorname{Teich}$ be the space of all complex structures such that η has type (1,1) on M.

a. Prove that $\dim_{\mathbb{R}} \operatorname{Teich}_\eta = \dim_{\mathbb{R}} \operatorname{Teich} - 2$

b. Use this calculation to prove that for each non-zero cohomology class $\eta \in H^2(M, \mathbb{Q})$ there exists a complex structure I on M such that $H^{1,1}(M, I) \cap H^2(M, \mathbb{Q})$ is generated by η.

c. Prove that such a torus (M, I) is equipped with an elliptic fibration $\pi : (M, I) \longrightarrow E$, where E is an elliptic curve, and all complex curves $C \subset (M, I)$ are fibers of π.

Hint. This is the same dimension count as in Exercise 1.12 (Handout 1).