VHS, home assignment: Polarized Hodge structures

Rules: This is a home assignment for the next week. We shall discuss the solutions next Tuesday.

3.1 Polarized Hodge structures of weight 1

Exercise 3.1. Prove that the category of integer Hodge structures of weight 1 is equivalent to the category of complex tori with fixed origin (a point), with objects compact complex tori and morphisms holomorphic homomorphisms preserving the origin.

Remark 3.1. Recall that an integer or rational Hodge structure is called "polarized" if it admits a polarization. The space of morphisms of polarized Hodge structures is the same as the space of their morphisms as Hodge structures.

Exercise 3.2. Let $(V_{\mathbb{Z}}, V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1})$ be an integer Hodge structure of weight 1, and $T = V^{1,0}/V_{\mathbb{Z}}$ the corresponding complex torus. Prove that T is projective if and only if the Hodge structure $(V_{\mathbb{Z}}, V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1})$ is polarized (here, the polarization is assumed to be rational).

Definition 3.1. Fix an anti-symmetric 2-form ω on a real vector space V. The period space Per of polarized Hodge structures on V is the set of all Hodge structures $(V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1})$ such that ω is a polarization on $V_{\mathbb{R}}$, that is, ω is I-invariant and satisfies $-\sqrt{-1}\omega(x, \bar{x}) > 0$ for all $x \in V^{1,0}$

Exercise 3.3. Suppose that the decomposition $(V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1})$ is associated with a complex structure operator I on $V_{\mathbb{R}}$. Prove that $-\sqrt{-1}\omega(x, \bar{x}) > 0$ for all non-zero $x \in V^{1,0}$ if and only if $\omega(v, Iv) > 0$ for all non-zero $v \in V_{\mathbb{R}}$.

Exercise 3.4. Prove that Per is the set of all complex structure operators $I \in \operatorname{End}(V_{\mathbb{R}})$ which satisfy $\omega(Iv, Iw) = \omega(v, w)$ and $\omega(v, Iv) > 0$ for all non-zero $v \in V_{\mathbb{R}}$.

Exercise 3.5. Prove that Per is an open subset of the Lagrangian grassmannian $\operatorname{Gr}_{lag}(V_{\mathbb{C}})$ consisting of all Lagrangian subspaces $L \subset (V_{\mathbb{C}}, \omega)$ which satisfy $-\sqrt{-1}\omega(x, \bar{x}) > 0$ for all $x \in L \backslash 0$.

Exercise 3.6.

a. Prove that $T_I \mathsf{Per}$ is the space of all matrices $A \in \operatorname{End}(V_{\mathbb{R}})$ which satisfy $\omega(Ax, y) = \omega(Ay, x)$ and $A(Iv) = -IA(v)$.

b. Fix $I \in \mathsf{Per}$, and let $A \in \operatorname{End}(V_{\mathbb{R}})$ and B be a 2-form on $V_{\mathbb{R}}$ which is given as $B(x, y) = \omega(Ax, y)$. Prove that $\omega(Ax, y) = \omega(Ay, x)$ and $A(Iv) = -IA(v)$ if and only if B is symmetric and satisfies $B(Ix, Iy) = -B(x, y)$.

Remark 3.2. A symmetric 2-form on $V_{\mathbb{R}}$ is called **anti-Hermitian** if it satisfies $B(Ix, Iy) = -B(x, y)$ for all $x, y \in V_{\mathbb{R}}$. We have just proven that $T_I \mathsf{Per}$ is the space of anti-Hermitian 2-forms.

Exercise 3.7. Prove that $\mathsf{Per} = Sp(2n)/U(n)$.

Exercise 3.8. Construct an $Sp(2n)$-invariant metric on Per.

Exercise 3.9. Let I be a complex structure operator on $V_{\mathbb{R}}$, and ω a Hermitian form, defining polarization on the corresponding Hodge structure. Consider the space $\mathcal{H} \subset \mathsf{Sym}^2(V_{\mathbb{R}}^*)$ of anti-Hermitian forms.

a. Prove that for all $A \in \mathcal{H}$, the form $A(I\cdot, \cdot)$ is also symmetric and anti-Hermitian. Prove that this operation defines an $Sp(2n)$-invariant complex structure on Per.

b. Consider an automorphism of $\mathsf{Per} = Sp(2n)/U(n)$ induced by conjugation with I. Prove that this map defines an isometry of Per, and acts as $-\operatorname{Id}$ on $T_I \mathsf{Per}$.

c. Prove that $\mathsf{Per} = Sp(2n)/U(n)$ admits a structure of a Hermitian symmetric space.

3.2 Localization in categories

Definition 3.2. Let $\mathcal{C}$ be a category and S a collection of morphisms. A category $\mathcal{C}_S$ together with a functor $\Psi : \mathcal{C} \longrightarrow \mathcal{C}_S$ is called **localization of $\mathcal{C}$ in S** if for all $s \in S$, the images $\Psi(s)$ are isomorphisms, and for any other functor $\Psi' : \mathcal{C} \longrightarrow \mathcal{C}'$ such that all images $\Psi'(s)$ are isomorphisms, Ψ' is factorized through Ψ.

Exercise 3.10. Let $\mathcal{C}$ be the category of finitely generated abelian groups, and S the set of all morphisms with finite kernel and finite index. Prove that the localization $\mathcal{C}_S$ is the category of all finite-dimensional vector spaces over $\mathbb{Q}$.

Exercise 3.11. Let $\mathcal{C}$ be a category with a unique object A. Prove that the functors $\Psi' : \mathcal{C} \longrightarrow \mathcal{C}'$ are in bijective correspondence with pairs

$$B \in \mathcal{O}b(\mathcal{C}'), \psi \in \operatorname{Hom}(\mathcal{M}or(A,A), \mathcal{M}or(B,B)),$$

where ψ is a morphism of semigroups.

Exercise 3.12. Let Ω be a semigroup, and Λ the group generated by the same generators and relations as Ω. Consider a category $\mathcal{C}$ with a unique object A and $\mathcal{M}or(A,A) = \Omega$. Prove that the localization of $\mathcal{C}$ in $\mathcal{M}or(A,A)$ is category $\mathcal{C}_1$ with a unique object B and $\mathcal{M}or(A,A) = \Lambda$.

Exercise 3.13. Let R be a ring, $\mathcal{C}$ the category of R-modules, $\mathfrak{S} \subset R$ a multiplicatively closed subset, not containing 0, and $S \subset \mathcal{M}or(\mathcal{C})$ the set of all morphisms $A \longrightarrow A$ given by a multiplication with $s \in \mathfrak{S}$. Denote by R_S the ring R localized in $\mathfrak{S}$. Prove that the localization of $\mathcal{C}$ in S is the category of R_S-modules.

Exercise 3.14. Let $\mathcal{C}$ be the category of integer Hodge structures, and S the set of morphisms $V_\mathbb{Z} \longrightarrow W_\mathbb{Z}$ between integer Hodge structures $(V_\mathbb{Z}, V_\mathbb{C} = \bigoplus_{p+q=w} V^{p,q})$ and $(W_\mathbb{Z}, W_\mathbb{C} = \bigoplus_{p+q=w} W^{p,q})$ which induces an isomorphism $V_\mathbb{C} \longrightarrow W_\mathbb{C}$. Prove that the localization $\mathcal{C}_S$ is the category of rational Hodge structures of weight 1.

Definition 3.3. An isogeny of complex tori is a holomorphic group homomorphism which is a finite covering. **A category of complex tori up to isogeny** is category of complex tori, localized in isogenies.

Exercise 3.15. Prove that the category of complex tori up to isogeny is equivalent to the category of rational Hodge structures.

Exercise 3.16. The category of abelian varieties is the full subcategory of the category of complex tori, with objects abelian varieties, that is, complex tori which are projective varieties. Prove that the category of polarized integer Hodge structures of weight 1 is equivalent to the category of abelian varieties.

Definition 3.4. An exact sequence $0 \to A \to B \to C \to 0$ **splits** if $B = A \oplus C$, and this decomposition is compatible with the arrows. An abelian category $\mathcal{C}$ is **semisimple** if any exact sequence in $\mathcal{C}$ splits.

Exercise 3.17. Prove that the category of polarized rational Hodge structures is semisimple.

Exercise 3.18. Let $A \subset B$ be abelian varieties. Prove that B contains an abelian subvariety C of complementary dimension, intersecting A transversally.

Hint. Use the previous exercise.

Remark 3.3. For complex tori this statement is false, as follows from the last exercise in assignment 2.