Variation of Hodge Structures, 2024: final exam

Rules: Every student gets 9 exercises (randomly chosen from this problem set), the final grade is determined by the score. Please write down the solutions and bring them to me at the exam time. To pass the exam, you are required to explain the solutions, using your notes. Please learn the proofs of all results you will be using on the way (you may put them in your notes). Please contact me in person or by email `verbit[]impa.br` when you are ready.

The final score N is obtained by summing up the points from the exam problems, adding 1 for each test exercise of the 4 sets of written exercises that you have been given. Marks: C when $20 \leqslant N < 40$, B when $40 \leqslant N < 50$, A when $50 \leqslant N < 80$, A+ when $N \geqslant 80$.

1 Rational Hodge structures

Exercise 1.1 (10 pt). Let $V = (V_{\mathbb{Q}}, V_{\mathbb{C}} = \bigoplus V^{p,q})$ be an irreducible Hodge structure, and $E \subset \operatorname{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$ its Hodge endomorphism algebra. Assume that V admits a polarization. Prove that the trace form $\operatorname{Tr} \in \mathsf{Sym}^2 E^*$ taking A, B to $\operatorname{Tr}(AB)$ is non-degenerate.

Definition 1.1. A Hodge structure of K3 type is a Hodge structure with $V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$, and with $\dim_{\mathbb{C}} V^{2,0} = 1$.

Exercise 1.2. Let $(V_{\mathbb{Q}}, V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2})$ be a Hodge structure of K3 type, which is irreducible with $\dim_{\mathbb{C}} V^{1,1} = 1$, and $K \subset \operatorname{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$ its endomorphism algebra.

a. (10 pt) Prove that K is degree 3 field, or $K = \mathbb{Q}$.

b. (30 pt) Prove that $K = \mathbb{Q}$ if V admits a polarization, or find a counterexample.

Exercise 1.3 (10 pt). Let $V_{\mathbb{Q}} = \mathbb{Q}^3$ and $A \in GL(V_{\mathbb{Q}})$ an endomorphism with irreducible characteristic polynomial $P(t)$. Assume that $P(t)$ has 2 roots in $\mathbb{C} \backslash \mathbb{R}$. Prove that A preserves a Hodge structure on V. Prove that this Hodge structure is unique up to a complex conjugation.

Exercise 1.4 (20 pt). Consider an irreducible Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}} = V^{3,0} \oplus V^{2,1} \oplus V^{1,2} \oplus V^{0,3})$ with all Hodge components of dimension 1. Prove that its endomorphism algebra is a number field. Construct an example of a Hodge structure of this type such that K is a degree 4 number field.

Exercise 1.5. Let $V_{\mathbb{Q}}$ be a rational vector space, $\dim_{\mathbb{Q}} V_{\mathbb{Q}} = 2n$, and $A \in \operatorname{End}(V_{\mathbb{Q}})$ an endomorphism such that its characteristic polynomial $P(t)$ is irreducible and has no real roots.

a. (10 pt) Prove that there exists a Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}} = \bigoplus V^{p,q})$ of odd weight with all $V^{p,q}$ 1-dimensional such that A preserves the Hodge decomposition.

b. (10 pt) Prove that there is a finite number such Hodge structures, if A is fixed.

c. (20 pt) Prove that such a Hodge structure is irreducible.

d. (20 pt) Suppose that the Hodge structure of this type is polarizable. Prove that this polarization is unique up to a constant or find a countexample. Prove that $t^{2n}P(t^{-1})$ is proportional to $P(t)$, if A preserves the polarization.

Exercise 1.6 (10 pt). Let $V_{\mathbb{Q}}$ be a rational vector space, $\dim_{\mathbb{Q}} V_{\mathbb{Q}} = 2n+1$, and $A \in \operatorname{End}(V_{\mathbb{Q}})$ an endomorphism such that its characteristic polynomial $P(t)$ is irreducible and has only one real root. Prove that there exists a Hodge structure $(V_{\mathbb{Q}}, V_{\mathbb{C}} = \bigoplus V^{p,q})$ of even weight with all $V^{p,q}$ 1-dimensional such that A preserves the Hodge decomposition. Prove that there are only finitely many such Hodge structures.

Exercise 1.7 (20 pt). Let $(V_{\mathbb{Q}}, V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2})$ be an irreducible, polarized Hodge structure of K3 type, $A \in \operatorname{End}_{\mathbb{Q}}(V_{\mathbb{Q}})$ its endomorphism, and $P(t)$ its characteristic polynomial. Assume that $P(t)$ is irreducible. Prove that $P(t)$ has at most 2 real roots.

Exercise 1.8 (10 pt). Let $(V_{\mathbb{Q}}, V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2})$ be an irreducible, polarized Hodge structure with $\dim_{\mathbb{C}} V^{2,0} = 2$, $\dim V^{1,1} = 1$. Prove that its endomorphism algebra is $\mathbb{Q}$.

Exercise 1.9 (30 pt). Let $(V_{\mathbb{Q}}, V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2})$ be an irreducible, polarized Hodge structure with $\dim_{\mathbb{C}} V^{2,0} = 2$, $\dim V^{1,1} = 2$. Prove that its endomorphism algebra is commuitative, or find a counterexample

2 Period spaces

Definition 2.1. Let $V = (V_{\mathbb{Q}}, V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q})$ be a Hodge structure. **The period space** for V is the set of all Hodge structures $(V_{\mathbb{C}} = \bigoplus_{p+q=w} V_1^{p,q})$ on the same complex space $V_{\mathbb{C}}$ with $\dim V^{p,q} = \dim V_1^{p,q}$. We denote this space by $\mathbb{P}\mathsf{er}(k_1, k_2, ..., k_{w+1})$ where k_i are dimensions of $V^{p,q}$ arranged from $V^{w,0}$ to $V^{0,w}$. When $V_{\mathbb{Q}}$ is equipped with a polarization form s, we also consider **the period space of polarized Hodge structures** which is the set of all is the set of all Hodge structures $V_1 = (V_{\mathbb{C}} = \bigoplus_{p+q=w} V_1^{p,q})$ such that s is a polarization on V_1. This space is denoted by $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(k_1, k_2, ..., k_{w+1})$.

Exercise 2.1 (10 pt). Prove that $\mathbb{P}\mathsf{er}(k_1, k_2, ..., k_{w+1})$ is an open subset in a complex flag space, and compute its complex dimension in terms of $k_1, ..., k_{w+1}$.

Exercise 2.2 (20 pt). Prove that $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(k_1, k_2, ..., k_{w+1})$ is a complex subvariety in an open subset of a complex flag space.

Exercise 2.3 (20 pt). Let $V_{\mathbb{R}} := V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. Prove that the group $SO(V_{\mathbb{R}}, s)$ acts on $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(k_1, k_2, ..., k_{w+1})$ transitively.

Exercise 2.4 (10 pt). Prove that $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,0,k,0,1)$ is diffeomorphic to the real Grassmannian $\operatorname{Gr}(2,k+2)$

Definition 2.2. Let W be a complex vector space equipped with a non-degenerate complex linear scalar product s, $\dim_{\mathbb{C}} W = 2n$. **The maximal isotropic Grassmannian** $\operatorname{GrIso}(W)$ is the space of all n-dimensional complex subspaces $L \subset W$ such that $s\Big|_L = 0$.

Exercise 2.5 (20 pt). Prove that $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(k,0,l,0,k)$ is a locally trivial fibration over a real Grassmannian $\operatorname{Gr}(2k,2k+l)$ with the fiber which is diffeomorphic to $\operatorname{GrIso}(\mathbb{C}^{2k})$.

Exercise 2.6 (10 pt). Prove that $\dim_{\mathbb{C}} \mathbb{P}\mathsf{er}(1,1,1,1) = 6$. Prove that $\dim_{\mathbb{C}} \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,1,1,1) = 4$.

Exercise 2.7 (10 pt). Prove that $\dim_{\mathbb{C}} \mathbb{P}\mathsf{er}^{\mathsf{pol}}(k,0,k) = \frac{k(k-1)}{2}$.

Exercise 2.8 (10 pt). Prove that $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(k,0,k)$ is diffeomorphic to $\operatorname{GrIso}(\mathbb{C}^{2k})$.

Exercise 2.9 (10 pt). Prove that $\mathbb{P}\mathsf{er}(1,1)$ is biholomorphic to $\mathbb{C}P^1 \backslash \mathbb{R}P^1$. Prove that $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,1)$ is biholomorphic to a disk.

Exercise 2.10 (30 pt). Prove that $\mathbb{P}\mathsf{er}(1,1,1)$ is holomorphically fibered over $\mathbb{C}P^2 \backslash \mathbb{R}P^2$ with fiber $\mathbb{C}$.

Exercise 2.11 (10 pt). Consider a holomorphic map $\mathbb{C} \longrightarrow \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,k,1)$. Prove that it is constant.

Exercise 2.12 (10 pt). Construct a non-constant holomorphic map $\mathbb{C} \longrightarrow \mathbb{P}\mathsf{er}(1,k,1)$, for all $k > 0$.

3 Variations of Hodge structures

Exercise 3.1 (20 pt). Let $X := \mathbb{P}\mathsf{er}(k_1, k_2, ..., k_{w+1})$ be the period space of Hodge structures, and $V = (V_{\mathbb{Q}}, V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q})$ a Hodge structure of the same type. Consider the space $\mathfrak{W} := \bigoplus_{p+q=w} \operatorname{Hom}\left(V^{p,q}, \bigoplus_{p',q' \neq p,q} V^{p',q'}\right)$, and let ι be an involution on this space which takes each $v \in \operatorname{Hom}\left(V^{p,q}, \bigoplus_{p',q' \neq p,q} V^{p',q'}\right)$ to its complex conjugate $\bar{v} \in \operatorname{Hom}\left(V^{q,p}, \bigoplus_{p',q' \neq q,p} V^{q',p'}\right)$. Prove that $T_V X = \{w \in W \ \mid \ \iota(w) = \iota\}$.

Definition 3.1. We use the notation introduced in the previous exercise. Define **the horizontal distribution** as a sub-bundle $T_{\mathsf{hor}}X \subset TX$

$$\bigoplus_{p+q=w} \operatorname{Hom}(V^{p,q}, V^{p-1,q+1} \oplus V^{p+1,q-1}) \subset \bigoplus_{p+q=w} \operatorname{Hom}\left(V^{p,q}, \bigoplus_{p',q' \neq p,q} V^{p',q'}\right) = T_V X.$$

Exercise 3.2 (20 pt). Let M be a simply connected complex manifold, B a trivial real bundle on M with the fiber $V_{\mathbb{R}}$, and $\nabla: B \longrightarrow B \otimes \Lambda^1 B$ its trivial connection. Let $\mathbb{P}\mathsf{er}$ be the period space for all real Hodge structures on $V_{\mathbb{R}}$. Prove that the set of all variations of Hodge structures defined on (B, ∇) is in bijective correspondence with the set of holomorphic maps $\phi: M \longrightarrow \mathbb{P}\mathsf{er}$ tangent to the horizontal distribution.

Exercise 3.3 (20 pt). Prove that the natural action of $GL(\mathbb{R}^{\sum k_i})$ on $\mathbb{P}\mathsf{er}(k_1, k_2, ..., k_{w+1})$ takes $T_{\mathsf{hor}}\, \mathbb{P}\mathsf{er}(k_1, k_2, ..., k_{w+1})$ to itself.

Exercise 3.4 (10 pt). Prove that $\dim T_{\mathsf{hor}}\, \mathbb{P}\mathsf{er}(1,1,1) = 1$.

Exercise 3.5 (20 pt). Prove that $T_{\mathsf{hor}}\, \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,k,1) = T\, \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,k,1)$. Prove that $T_{\mathsf{hor}}\, \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,k,1) \subsetneq T\, \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,k,1)$.

Definition 3.2. Let M be a complex manifold, and $\tilde{M}$ its universal cover. A variation $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ of Hodge structures over M is called **isotrivial** if the corresponding map $\tilde{M} \longrightarrow \mathbb{P}\mathsf{er}(k_1, k_2, ..., k_{w+1})$ is constant.

Exercise 3.6 (20 pt). Construct an isotrivial variation of Hodge structures of weight 1 such that the corresponding monodromy representation is ireducible.

Exercise 3.7 (10 pt). Construct an isotrivial variation of Hodge structures of weight 1 which is irreducible as a variation of Hodge structures, but its monodromy representation is not ireducible.

Exercise 3.8 (20 pt). Let $(B, \nabla, \bigoplus_{p+q=w} B^{p,q})$ be an isotrivial polarized variation of integer Hodge structures. Prove that its monodromy is finite.

Exercise 3.9 (20 pt). Let $(B, \nabla, B \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=w} B^{p,q})$ be a variation of Hodge structures over a simply connected manifold M, η a parallel section of B, and M_η the set of all points $x \in M$ such that $\eta\Big|_x$ belongs to $B^{p,q} \oplus B^{p+1,q-1} \oplus ... \oplus B^{q,p}$ for some numbers p, q such that $p > q$. Prove that $M_\eta \subset M$ is complex analytic.

Exercise 3.10 (30 pt). Prove that $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,0,1,0,1)$ is biholomorphic to $\mathbb{C}P^1$. Consider $\mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,0,1,0,1)$ as the space of real Hodge structures on a vector space $V = \mathbb{R}^3$. Fix $v \in V$. Consider the space $M_v \subset \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,0,1,0,1)$ defined as the set of all Hodge structures such that $v \in V^{2,2}$, and $M'_v \subset \mathbb{P}\mathsf{er}^{\mathsf{pol}}(1,0,1,0,1)$ the set of all Hodge structures such that $v \in V^{4,0} \oplus V^{0,4}$. Prove that M_v is 2 points and M'_v is a circle.