Variations of Hodge structures

lecture 1: Hodge structures

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Hodge structures

DEFINITION: Let $V_{\mathbb{R}}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb{Q}}$ such that $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes \mathbb{R}$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

REMARK: **Rational structure** on a real vector space V is a $\mathbb{Q}$-subspace $V_{\mathbb{Q}} \subset V$ such that $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. **Integer structure** on a real vector space V is a $\mathbb{Z}$-sublattice $V_{\mathbb{Z}} \subset V$ such that $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$.

REMARK: A real Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$ on $V_{\mathbb{R}}$ is rational (integer) **if $V_{\mathbb{R}}$ is equipped with a rational (integer) structure.**

Polarization

DEFINITION: **Polarization** on a rational Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$-\sqrt{-1}^{\,p-q} h(x, \overline{x}) > 0 \qquad (*)$$

("Hodge-Riemann relations") for each non-zero $x \in V^{p,q}$.

DEFINITION: The objects of the **category of real (rational, integer) Hodge structures** are Hodge structures, morphisms are $\mathbb{R}$-linear ($\mathbb{Q}$-linear, $\mathbb{Z}$-linear) maps of vector spaces which preserve the Hodge decomposition.

DEFINITION: The objects of the **category of rational or integer polarized Hodge structures are rational or integer Hodge structures admitting a polarization**, and morphisms are morphisms of Hodge structures. **The polarization is not fixed, the morphisms are not necessarily compatible with the polarization.**

The Hodge decomposition on a Grassmann algebra

DEFINITION: The Hodge decomposition $V\otimes_{\mathbb R}\mathbb C := V^{1,0}\oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb C} := V\otimes_{\mathbb R}\mathbb C$. The Grassmann algebra of skew-symmetric forms $\Lambda^* V_{\mathbb C} := \Lambda^*_{\mathbb R} V\otimes_{\mathbb R} C$ admits a decomposition

$$\Lambda^m V_{\mathbb C} = \bigoplus_{p+q=m}\Lambda^p V^{1,0}\otimes\Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0}\otimes\Lambda^q V^{0,1}$ by $\Lambda^{p,q}V$. The resulting decomposition $\Lambda^m V_{\mathbb C} = \bigoplus_{p+q=m}\Lambda^{p,q}V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The decomposition $\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n}\Lambda^{p,q}V$ **defines a Hodge structure on $\Lambda^m V$.** If, in addition, V was equipped with a rational or an integer structure, **the space $\Lambda^m V$ inherits a rational (integer) Hodge structure.**

$U(1)$-representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}\,pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$.**

REMARK: In other words, **the category of graded vector spaces is equivalent to the category of vector spaces equipped with a $U(1)$-action.**

REMARK: The operator I induces $U(1)$-action on V by the formula $\rho(t)(v) = \cos t \cdot v + \sin t \cdot I(v)$. We extend this action on the tensor spaces by muptiplicativity.

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition, with $\Lambda^{p,q}V$ being a weight $p-q$-component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$-invariant vectors in $\Lambda^{2p}V$.

Adjoint operators

REMARK: Let V be a vector space equipped with a positive definite scalar product. **Then all tensor powers of V and its Grassmann algebra is also equipped with a positive definite scalar product, using the standard formula.**

REMARK: Let (M, g) be an oriented Riemannian manifold. Denote by Vol_g the Riemannian volume form. The natural $C^\infty M$-valued pairing on differential forms is also denoted by g. Consider a differential operator $\delta : \Lambda^* M \longrightarrow \Lambda^* M$. An operator $\delta^* : \Lambda^* M \longrightarrow \Lambda^* M$ is called **adjoint operator to δ** if $\int_M g(\delta^* a, \tau) \operatorname{Vol}_g = \int_M g(a, \delta\tau) \operatorname{Vol}_g$ for any form $\tau \in \Lambda^* M$ with compact support.

CLAIM: An adjoint operator δ^* **exists for any differential operator $\delta : \Lambda^* M \longrightarrow \Lambda^* M$.** Moreover, **$\delta^*$ is also a differential operator.**

Proof: http://verbit.ru/MATH/Hodge-2018/slides-Hodge-01.pdf and http://verbit.ru/MATH/Hodge-2018/listok-Hodge-06.pdf. ■

Hodge theory for Riemannian manifolds

DEFINITION: Let M be a Riemannian manifold, and $d: \Lambda^* M \longrightarrow \Lambda^{*+1} M$ the de Rham differential. Denote by $d^*: \Lambda^* M \longrightarrow \Lambda^{*-1} M$ its adjoint operator. The operator $\Delta := dd^* + d^*d$ is called **the Laplace operator.** A differential form η is called **harmonic** if $\eta \in \ker \Delta$.

REMARK: Let M be a compact Riemannian manifold, and η a harmonic form. Then $g(dd^*\eta + d^*d\eta, \eta) = g(d\eta, d\eta) + g(d^*\eta, d^*\eta)$, hence η is harmonic if and only if $d\eta = d^*\eta = 0$. Also, by definition, $d^*\eta = 0$ if and only if $\eta \perp \operatorname{im} d$. **This implies that all harmonic forms are closed and orthogonal to exact, producing an embedding from the space of harmonic forms to de Rham cohomology.**

THEOREM: (the main theorem of Hodge theory)
Let M be a compact Riemannian manifold. Then **the natural embedding from the space of harmonic forms to de Rham cohomology is an isomorphism.**

Proof: http://verbit.ru/MATH/Hodge-2018/slides-Hodge-08.pdf ■

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

"The usual definition": A complex manifold **is a smooth manifold with an atlas of open balls $\{U_i\}$, each U_i diffeomorphic to $B \subset \mathbb{C}^n$ and the transition functions complex analytic.**

THEOREM: (Newlander-Nirenberg)
These two definitions are equivalent.

Kähler manifolds

DEFINITION: A Hermitian metric on an almost complex manifold is a Riemannian metric g such that $g(Ix, Iy) = g(x, y)$. The corresponding **Hermitian form** is $\omega(x, y) := g(Ix, y)$, which is non-degenerate and antisymmetric.

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

Proof: `http://verbit.ru/MATH/KAHLER-2020/slides-Kahler-2020-10.pdf` ■

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Hodge structures on cohomology of a Kähler manifold

THEOREM: Let M be a compact Kähler manifold, and Δ its Laplacian operator. **Then Δ preserves the Hodge decomposition** (that is, commutes with the corresponding $U(1)$-action). Moreover, the operator of orthogonal projection to $\ker\Delta$ **also commutes with the $U(1)$-action**, and maps a closed form to a cohomologous harmonic form.

Proof: http://verbit.ru/MATH/KAHLER-2020/slides-Kahler-2020-12.pdf ■

COROLLARY: Let M be a compact Kähler manifold. Then its de Rham cohomology $H^*(M,\mathbb{R})$ **is equipped with the natural Hodge structure.** Moreover, for any holomorphic map $f: M \longrightarrow N$ of compact Kähler manifolds, the corresponding action on cohomology $f^*: H^*(M) \longrightarrow H^*(M)$ **defines a morphism of Hodge structures.**

Proof. Step 1: Consider the standard $U(1)$-action on the Grassmann algebra, $\rho(t)(v) = \cos t\cdot v + \sin t\cdot I(v)$. This action commutes with the Laplacian, **hence it is defined on the cohomology.** It defines the Hodge decomposition $H^m(M,\mathbb{C}) = \bigoplus_{p+q=m} H^{p,q}(M)$.

Step 2: A map f is holomorphic if and only if $f^*: \Lambda^*(N) \longrightarrow \Lambda^*(M)$ commutes with the $U(1)$-action. **Therefore, f^* preserves the Hodge decomposition on the differential forms,** and hence on the cohomology. ■

Tensor product of Hodge structures

DEFINITION: Let V, W be vector spaces equipped with a Hodge structure (real, rational or integer). The tensor product $V \otimes_{\mathbb R} W$ **is also equipped with a Hodge structure** (real, rational or integer), as follows. The $U(1)$-action on $V \otimes_{\mathbb R} W$ is obtained by taking a tensor product of $U(1)$-representations: $\rho(t)(v \otimes w) = \rho_V(t)(v) \otimes \rho_W(t)(w)$. **This means that the corresponding Hodge components are multiplied via**

$$(V \otimes_{\mathbb R} W)^{p,q} = \bigoplus_{p_1+p_2=p, q_1+q_2=q} V^{p_1,q_1} \otimes W^{p_1,q_2}.$$

The rational or integer structure on $V \otimes_{\mathbb R} W$ is obtained as $(V \otimes_{\mathbb R} W)_{\mathbb Q} = V_{\mathbb Q} \otimes_{\mathbb Q} W_{\mathbb Q}$ and $(V \otimes_{\mathbb R} W)_{\mathbb Z} = V_{\mathbb Z} \otimes_{\mathbb Z} W_{\mathbb Z}$.

REMARK: Given polarizations t, s on V, W, **we obtain a polarization $t \otimes s$ on $V \otimes W$,** defined using $t \otimes s(v \otimes w, v' \otimes w') = t(v,v')s(w,w')$.

PROPOSITION: Let M_1, M_2 be compact Kähler manifolds, and $M := M_1 \otimes M_2$. Then the Künneth formula $H^*(M) = H^*(M_1) \otimes H^*(M_2)$ **is compatible with the tensor product of Hodge structures.**

Proof: The natural map $\Lambda^p M_1 \otimes \Lambda^q M_2 \longrightarrow \Lambda^{p+q}(M)$ is compatible with the tensor product, and takes a product of harmonic forms to a harmonic form.
■

Representations of $\mathfrak{sl}(2)$: the highest vector

REMARK: We have defined the Hodge structure on cohomology of any compact Kähler manifold M. However, to define the polarization, M should be projective. Also, the polarized Hodge structure is defined on the "primitive part of the cohomology" only. **To define the "primitive part of the cohomology", we use the Lefschetz $\mathfrak{sl}(2)$-action.**

DEFINITION: Let $\mathfrak{b}^+ \subset \mathfrak{sl}(2)$ be the **upper triangular Borel subalgebra,** that is, the 2-dimensional algebra of upper triangular matrices.

DEFINITION: Let V be a representation of $\mathfrak{sl}(2)$ **A highest vector** $v \in V$ is a vector on which the nilpotent part of $\mathfrak{b}^+$ acts trivially. and the diagonal part $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as a multiplication by a constant.

REMARK: Prove that any finite-dimensional representation of $\mathfrak{sl}(2)$ **contains a highest vector.**

Fundamental representation of $\mathfrak{sl}(2)$ and its symmetric powers

DEFINITION: The fundamental representation of $\mathfrak{sl}(2)$ is the standard 2-dimensional representation.

REMARK: Prove that **a symmetric power $V_p = \operatorname{Sym}^p(F)$ of the fundamental representation is irreducible.** The representation V_p is called **an irreducible representation of weight p.**

REMARK: Let V be a finite-dimensional irreducible representation of $\mathfrak{sl}(2)$. Prove that the highest vector of V **is unique up to a constant.** Prove that V **is isomorphic to a symmetric power of the fundamental representation.**

REMARK: Consider the action of $H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on V_p. **Prove that** $\dim V_p = p+1$ **and H acts as diagonal martix with eigenvalues $-p, -p+2, ..., p-2, p$.**

The Lefschetz $\mathfrak{sl}(2)$-action

DEFINITION: Let (M, I, ω) be a Kähler manifold, $\dim_{\mathbb{C}} M = n$ Denote by $L: \Lambda^*(M) \longrightarrow \Lambda^{*+2}(M)$ the operator $\eta \mapsto \eta \wedge \omega$, by $\Lambda := L^*$, its adjoint, with $\Lambda: \Lambda^*(M) \longrightarrow \Lambda^{*-2}(M)$, and by H the commutator, $H := [L, \Lambda]$.

PROPOSITION: The operators L, Λ, H **form a basis of a Lie algebra isomorphic to $\mathfrak{sl}(2)$,** with relations

$$[L, \Lambda] = H, \quad [H, L] = 2L, \quad [H, \Lambda] = -2\Lambda.$$

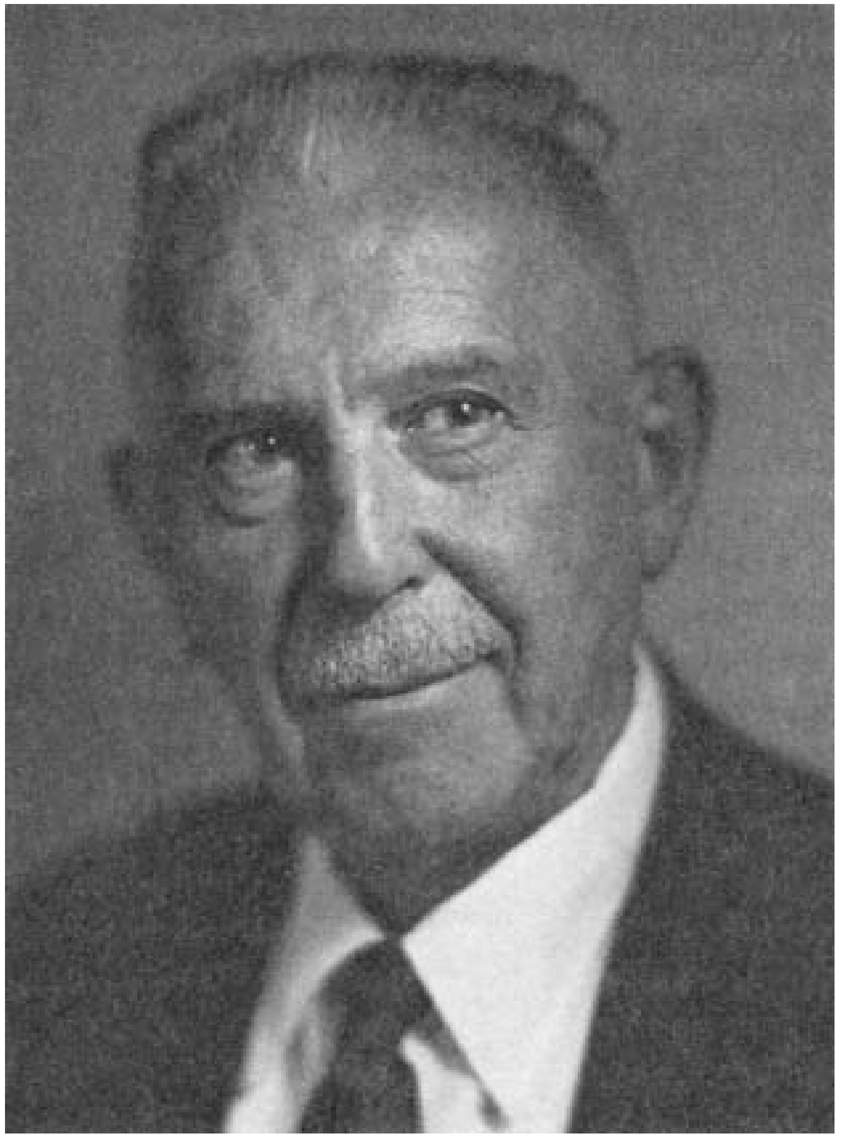
and H is a scalar operator acting on $\Lambda^p(M)$ as $(n-p)\operatorname{Id}$.

Proof: http://verbit.ru/MATH/KAHLER-2020/slides-Kahler-2020-12.pdf ■

W. V. D. Hodge (1903-1975)

Sir William Vallance Douglas Hodge FRS
(17 June 1903, Edinburgh - 7 July 1975, Cambridge)

Solomon Lefschetz (1884-1972)



Solomon Lefschetz (1884-1972)

The Hodge-Riemann bilinear relations

DEFINITION: Let M be a compact Kähler manifold, and $p+q \leqslant n = \dim_{\mathbb{C}} M$. Denote by $H^*(M) = \bigoplus_{r=0}^n W_r$ the weight decomposition associated with the Lefschetz $\mathfrak{sl}(2)$-action, with W_r a direct sum of irreducible $\mathfrak{sl}(2)$-representations of weight r. Let $V_k^{p,q} := H^{p,q}(M) \cap W_k$. **Clearly,** $H^*(M) = \bigoplus_{k,p,q} V_k^{p,q}$. **The Riemann-Hodge form** on $V_k^{p,q}$ is

$$\eta, \eta' \longrightarrow \sqrt{-1}^{p-q}(-1)^{p+q-k} \int_M \eta \wedge \overline{\eta}' \wedge \omega^{n-p-q}$$

THEOREM: (Riemann-Hodge relations)
The Riemann-Hodge form is positive definite.

Proof: Follows from Weyl's structure theorem on tensor representations of $U(n)$. See *Howe, Roger E., "Remarks on classical invariant theory", Transactions of the American Mathematical Society, American Mathematical Society, 313 (2): 539-570, 1989.* ■

The Hodge-Riemann bilinear relations and the polarization

COROLLARY: Let M be a compact projective manifold, and $W_k^m \subset H^m(M)$ be the space of all (p,q)-classes, $p+q=m$, which belong to an $\mathfrak{sl}(2)$-representation of weight k. **Then W_k^m is a Hodge structure, and the Riemann-Hodge pairing, multiplied by ± 1, defines a polarization on W_k^m.**

Proof. Step 1: Since M is projective, the cohomology class of ω is integer. Since two elements H and L of the Lefschetz triple are rational, the third, Λ is also rational. This implies that the decomposition $H^m(M) = \bigoplus_k W_k^m$ is rational; also, it is $U(1)$-invariant, and therefore **each W_k^m is a Hodge substructure in $H^m(M)$.** It remains only to show that it is polarized.

Step 2: Consider the bilinear form $q(\eta,\eta') := \int_M L^{n-p}(\eta\wedge\eta')$ for $\eta \in H^p(M)$, $p \leqslant \dim_{\mathbb{C}} M$ and $q(\eta,\eta') := \int_M \Lambda^{p-n}(\eta\wedge\eta')$ for $p \leqslant \dim_{\mathbb{C}} M$. This form is rational, because the operators L and Λ are rational. **From the Hodge-Riemann bilinear relations it follows that $\pm q$ also defines a polarization.**

■

Primitive and coprimitive classes

DEFINITION: A cohomology class η is called **primitive** if $\Lambda(\eta) = 0$, and **coprimitive** if $L(\eta) = 0$.

COROLLARY: The pairing $q(\eta, \eta') := \int_M L^{n-p}(\eta \wedge \eta')$ **defines a polarization on primitive classes.** The pairing $q(\eta, \eta') := \int_M \Lambda^{p-}(\eta \wedge \eta')$ **defines a polarization on coprimitive classes.**

Proof: Primitive classes are lowest vectors in an $\mathfrak{sl}(2)$-representation, hence the set of primitive classes is equal to W^p_{n-p}. Coprimitive classes are highest vectors in an $\mathfrak{sl}(2)$-representation, hence the set of coprimitive classes is equal to W^p_{p-n}. ■

DEFINITION: A **simple object** of an abelian category is an object which has no proper subobjects. An abelian category is **semisimple** if any object is a direct sum of simple objects.

CLAIM: Category of polarized Hodge structures in semisimple.

Proof: Orthogonal complement of a Hodge substructure $V' \subset V$ with respect to h is again a Hodge substructure, and this complement does not intersect V'; both assertions follow from the Hodge-Riemann relations. ■

Hodge structures of weight 1

DEFINITION: **An abelian variety** is a complex projective variety with a marked point which is biholomorphic to a compact torus $\mathbb{C}^n/\mathbb{Z}^{2n}$.

DEFINITION: Two complex tori T_1, T_2 are called **isogeneous** if there exists a surjective finite holomorphic map $T_1 \longrightarrow T_2$.

REMARK: The category of complex tori with marked point **is equivalent to the category of integer Hodge structures of weight 1**. The category of complex tori up to isogeny **is equivalent to the category of rational Hodge structures of weight 1.**

REMARK: Under this correspondence, **abelian varieties correspond to Hodge structures admitting a polarization.**

CLAIM: **The category $\mathcal{C}$ of rational Hodge structures admitting a polarization is semisimple,** that is, any object of $\mathcal{C}$ is a direct sum of irreducible ones.

REMARK: In particular, **the category of abelian varieties up to isogeny is semisimple.**