Variations of Hodge structures

lecture 2: Period maps and Torelli theorems

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Hodge structures (reminder)

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight w** on a vector space $V_{\mathbb C} = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb Q}$ such that $V_{\mathbb R} = V_{\mathbb Q} \otimes \mathbb R$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb Z} \subset V_{\mathbb Q}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

REMARK: **Rational structure** on a real vector space V is a $\mathbb Q$-subspace $V_{\mathbb Q} \subset V$ such that $V = V_{\mathbb Q} \otimes_{\mathbb Q} \mathbb R$. **Integer structure** on a real vector space V is a $\mathbb Z$-sublattice $V_{\mathbb Z} \subset V$ such that $V = V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$.

REMARK: A real Hodge structure $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ on $V_{\mathbb R}$ is rational (integer) **if $V_{\mathbb R}$ is equipped with a rational (integer) structure.**

Period space

DEFINITION: Let V be a real vector space, and $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure. Assume that $p, q \geqslant r$ **The Hodge filtration** is the following filtration on the vector space $V_{\mathbb{C}}$:

$$0 \subset V^{r,w-r} \subset V^{r,w-r} \oplus V^{r+1,w-r-1} \subset V^{r,w-r} \oplus V^{r+1,w-r-1} \oplus V^{r+2,w-r-2} \oplus ...$$

Denote by F_n the n-th term of this filtration, $F_n := \bigoplus_{i=0}^{n-1} V^{r+i,w-r-i}$. Clearly, $V^{p,w-p} = F_{p-r+1} \cap \overline{F}_{w-p-r+1}$ **(prove this)**, hence **the Hodge filtration determines the Hodge structure uniquely.**

REMARK: Two subspaces $W_1, W_2 \subset V$ intersect transversally when $W_1 + W_2 = V$. **Therefore, F_{p-r+1} and $\overline{F}_{w-p-r+1}$ intersect transversally.**

DEFINITION: Let $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure on V. Fix dimensions of all $V^{p,q}$; this determines the dimensions of F_i. **The period space** is the space of all flags $0 \subset F_1 \subset ... \subset F_{w-2r} = V$ such that F_{p-r+1} and $\overline{F}_{w-p-r+1}$ intersects transversally for all p.

CLAIM: The points in the period space **are in bijective correspondence with the set of all Hodge structures on V** having the same numbers $\dim V^{p,w-p}$. ■

REMARK: The period space **is an open subset in the corresponding partial flag space**, which is considered as a complex projective manifold.

Polarization (reminder)

DEFINITION: Polarization on a rational Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$-\sqrt{-1}^{\,p-q} h(x, \overline{x}) > 0 \qquad (*)$$

("Hodge-Riemann relations") for each non-zero $x \in V^{p,q}$.

REMARK: $U(1)$-invariance of a pairing h means that the pairing of $V^{p,q}$ with V^{p_1,q_1} vanishes unless $p - q = q_1 - p_1$, or, equivalently, $p = q_1, q = p_1$.

DEFINITION: The objects of the **category of real (rational, integer) Hodge structures** are Hodge structures, morphisms are $\mathbb{R}$-linear ($\mathbb{Q}$-linear, $\mathbb{Z}$-linear) maps of vector spaces which preserve the Hodge decomposition.

DEFINITION: The objects of the **category of rational or integer polarized Hodge structures are rational or integer Hodge structures admitting a polarization**, and morphisms are morphisms of Hodge structures. **The polarization is not fixed, the morphisms are not necessarily compatible with the polarization.**

Period space for polarized Hodge structures

DEFINITION: Let V be a real vector space, and $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure. Fix a polarization $h \in V^* \otimes V^*$ (symmetric or antisymmetric depending on parity of w). **The period space** of Hodge structures with polarization h is the space of all flags $0 \subset F_1 \subset ... \subset F_{w-2r} = V$ such that F_{p-r+1} and $\overline{F}_{w-p-r+1}$ intersects transversally for all p, and h restricted to this intersection satisfies the Hodge-Riemann condition $-\sqrt{-1}^{2p-w} h(x, \overline{x}) > 0$, for all non-zero $x \in F_{p-r+1} \cap \overline{F}_{w-p-r+1}$, and the pairing of all other Hodge components vanishes.

The Hodge-Riemann condition is open, but the condition "pairing of all other Hodge components vanishes" is closed. However, it can be formulated in terms of the Hodge filtration.

CLAIM: Let $h \in V^* \otimes V^*$ be a pairing. **Then h vanishes on all Hodge components except $V^{p,q} \times V^{q,p}$ if and only if $F_p^\perp = \overline{F}_{w-p-r}$.**

Proof: Left as an exercise. ■

Complex manifolds (reminder)

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

"The usual definition": A complex manifold **is a smooth manifold with an atlas of open balls $\{U_i\}$, each U_i diffeomorphic to $B \subset \mathbb{C}^n$ and the transition functions complex analytic.**

THEOREM: (Newlander-Nirenberg)
These two definitions are equivalent.

Kähler manifolds (reminder)

DEFINITION: A Hermitian metric on an almost complex manifold is a Riemannian metric g such that $g(Ix, Iy) = g(x, y)$. The corresponding **Hermitian form** is $\omega(x, y) := g(Ix, y)$, which is non-degenerate and antisymmetric.

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

Proof: http://verbit.ru/MATH/KAHLER-2020/slides-Kahler-2020-10.pdf ■

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Teichmüller space

DEFINITION: The space of almost complex structures is an infinite-dimensional Fréchet manifold X_M of all tensors $I^2 = -\operatorname{Id}_{TM}$, equipped with the natural Fréchet topology.

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let Teich := Comp / $\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: The space of $\operatorname{Diff}_0(M)$-orbits in a small neighbourhood of a point in Comp is always **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady). However, the quotient Comp / $\operatorname{Diff}_0(M)$ **is often non-Hausdorff**.

DEFINITION: We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**.

REMARK: The topology of the space Teich $/\Gamma$ is often bizzarre. However, **its points are in bijective correspondence with equivalence classes of complex structures.**

Kodaira-Spencer stability theorem

REMARK: A complex structure I on M is called **of Kähler type** if (M, I) admits a Kähler metric.

THEOREM: (**Kodaira-Spencer stability theorem**)
Let B be a manifold and $I_t, t \in B$ be a smooth family of complex structures on a compact manifold M parametrized by $t \in B$. Denote by $B_{\subset} B$ the set of all $t \in B$ such that I_t is of Kähler type. **Then B_0 is open in B.**

Proof: http://verbit.ru/MATH/LCK-2014/lck-07.pdf, page 18. ■

REMARK: From now on, speaking of Teichmüller spaces, we will always tacitly consider the Teichmüller space of complex structures **of Kähler type.** By Kodaira-Spencer stability theorem, **this space is open in the Teichmüller space of all complex structures.**

Period map

Let M_t be a smooth family of compact complex manifolds of Kähler type, $t \in \mathbb{R}$. Using degeneration of the Hodge to de Rham spectral sequence and semicontinuity of cohomology, we obtain the following

PROPOSITION: Let M_t be a smooth family of compact complex manifolds of Kähler type, $t \in \mathbb{R}$. **Then the numbers** $\dim H^{p,q}(M_t)$ **are constant in** t**.**

Proof: *S. R. Bell, R. Narasimhan, Proper holomorphic mappings of complex spaces, Encyclopaedia Math. Sci. 69, 1990.* ■

This implies that the Hodge structure on $H^*(M_t)$ has the same combinatorial type and the same period space.

DEFINITION: Let (M, I) be a compact complex manifold of Kähler type, and $\operatorname{Teich} \ni I$ the corresponding connected component of the space of complex structures of Kähler type on M. With each $I_1 \in \operatorname{Teich}$ we can associate the Hodge structure $\bigoplus_{p+q=w} H^{p,q}(M, I_1)$ on cohomology. This defines **the period map** $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$, where $\mathbb{P}\mathrm{er}$ is the period space of Hodge structures of this particular combinatorial type on $H^w(M, \mathbb{R})$.

“Torelli theorems”

REMARK: A theorem of Torelli type is a result claiming that $\operatorname{Per} : \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ is locally an embedding, or locally a diffeomorphism, or an embedding globally, or a diffeomorphism globally.

The name “Torelli theorem” is due to André Weil, *André Weil (1957), ”Zum Beweis des Torellischen Satzes”, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. IIa: 32-53.*

Chez les Weil. André et Simone

André Weil: 6 May 1906 - 6 August 1998.

"Simone et André à Penthiévre, 1918-1919"

Torelli-type theorems: torus, Riemann surface

THEOREM: Let M be a compact torus, and $\mathsf{Per} : \text{Teich} \longrightarrow \mathbb{P}\text{er}$ the period map associated with $H^1(M)$. **Then Per is a diffeomorphism**.

THEOREM: Let M be a compact 1-dimensional complex manifold (that is, a Riemann surface), and $\mathsf{Per} : \text{Teich} \longrightarrow \mathbb{P}\text{er}$ the period map associated with $H^1(M)$. **Then Per is a closed embedding**.

Pseudopolarizations

DEFINITION: Let V be a vector space, and $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure. A bilinear form $h \in V^* \otimes_{\mathbb{R}} V^*$ is called **a pseudo-polarization** if it is non-degenerate, $U(1)$-invariant, symmetric for w even and anti-symmetric for w odd.

REMARK: **$U(1)$-invariance means that $q(V^{p,q}, V^{p_1,q_1}) = 0$ unless $p = q_1$ and $q = p_1$.** Therefore, **$U(1)$-invariance of h is equivalent to $F_p^\perp = \overline{F}_{w-p-r}$,** where F_i is the Hodge filtration defined above.

DEFINITION: Let V be a real space equipped with a Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$, and h a pseudopolarization. **The period space $\mathbb{P}\mathrm{er}_h$** of pseudo-polarized Hodge structures is the space of all filtrations $0 \subset F_1 \subset ... \subset F_{2w-2r}$ such that $V_{\mathbb{C}} = \bigoplus_p F_p \cap \overline{F}_{w-p-r+1}$ and $F_p^\perp = \overline{F}_{w-p-r}$.

REMARK: Note that $F_p^\perp = \overline{F}_{w-p-r}$ implies that the filtration F_i is determined by the first $w-r$ terms. We identify $\mathbb{P}\mathrm{er}_h$ with an open subset of the partial flag space $0 \subset F_1 \subset ... \subset \overline{F}_{w-r} \subset V$ with F_i the same dimension, and such that for all p the spaces F_p, $\overline{F}_{w-p-r+1}$ intersect transversally, giving $V_{\mathbb{C}} = \bigoplus_p F_p \cap \overline{F}_{w-p-r+1}$; here F_p for $p > w-r$ are recovered from $F_p^\perp = \overline{F}_{w-p-r}$.

REMARK: **$\mathbb{P}\mathrm{er}_h$ is equipped with a structure of a complex manifold,** because it is an open subset in the space of partial flags, which is projective.

The Hausdorff reduction

DEFINITION: Two points x, y in a topological space M are called **non-separable** if any open neighbourhoods of x and y intersect. Denote by $\sim$ an equivalence relation generated by non-separability. **Hausdorff reduction** of M is $M/\sim$ with the quotient topology.

A caution: The Hausdorff reduction **does not need to be Hausdorff.**

REMARK: Clearly, the period map $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ is factorized through the Hausdorff reduction map $\operatorname{Teich} \longrightarrow \operatorname{Teich}/\sim$. Indeed, **any continuous map to a Hausdorff space is factorized through the Hausdorff reduction map.**

A global Torelli theorem for K3 surfaces

DEFINITION: A K3 surface is a Kähler complex compact 2-dimensional manifold M with $b_1 = 0$ and $c_1(M, \mathbb{Z}) = 0$.

Let $\mathbb{P}\mathrm{er}_q$ be the pseudopolarized period space for the Hodge structure on $H^2(M)$, where q denotes the Poincare pairing.

THEOREM: Let M be a K3 surface, and Teich a connected component of its Teichmüller space. Then **non-separability is an equivalence relation, and for any non-separable $I_1, I_2 \in \mathrm{Teich}$, the corresponding K3 surfaces are isomorphic.** Moreover, **the period map $\mathrm{Per}: \ \mathrm{Teich}/\sim \longrightarrow \mathbb{P}\mathrm{er}_q$ associated with $H^2(M)$ is a homeomorphism.**

Proof: https://arxiv.org/abs/0908.4121 ■

Hyperkähler manifolds

DEFINITION: A complex manifold is called **holomorphically symplectic** if it is equipped with a non-degenerate, closed $(2,0)$-form.

DEFINITION: For the present purposes, **hyperkähler manifold of maximal holonomy** is a compact, holomorphically symplectic manifold M of Kähler type which satisfies $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

REMARK: It is not very hard to see that any smooth Kähler-type deformation of a hyperkähler manifold of maximal holonomy is again a hyperkähler manifold of maximal holonomy. Let Teich be a connected component of the Teichmüller space of Kähler-type deformations of the complex structure on a hyperkähler manifold of maximal holonomy. Then **all points of** Teich **correspond to holomorphically symplectic deformations of M which also have maximal holonomy.**

A global Torelli theorem for hyperkähler manifolds

CLAIM: Let M be a hyperkähler manifold of maximal holonomy. Then $H^2(M)$ is equipped with a Diff(M)-invariant form q, called **Bogomolov-Beauville-Fujiki form** (BBF form), defining a pseudo-polarization on $H^2(M)$ for any complex structure of hyperkähler type.

Denote by $\mathbb{P}\mathrm{er}_q$ the pseudopolarized period space for the Hodge structure on $H^2(M)$ and the BBF form q.

THEOREM: Let M be a hyperkähler manifold of maximal holonomy, and Teich its Teichmüller space. Then **non-separability is an equivalence relation, and for any non-separable $I_1, I_2 \in$ Teich, the corresponding complex manifolds are bimeromorphic.** Moreover, **the period map Per : Teich $/ \sim \longrightarrow \mathbb{P}\mathrm{er}_q$ associated with $H^2(M)$ is a homeomorphism.**

Proof: https://arxiv.org/abs/0908.4121 ■

4-dimensional cubic hypersurface

For all results about 4-dimensional cubics, the universal reference is "The geometry of cubic hypersurfaces" by D. Huybrechs, https://www.math.uni-bonn.de/people/huybrech/Notes.pdf

PROPOSITION: Let M be a smooth cubic hypersurface in $\mathbb{P}^5$. **Then $H^4(M)$ has Hodge structure with $\dim H^{1,3}(M) = \dim H^{3,1}(M) = 1$ and $\dim H^{2,2}(M) = 20$.**

THEOREM: A Kähler deformation of a 4-dimensional cubic hypersurface is a 4-dimensional cubic hypersurface.

Proof: Huybrechts, Corollary 3.14. ■

This defines a period map $\mathsf{Per}: \text{Teich} \longrightarrow \mathbb{P}\mathrm{er}$, associated with the Hodge structure on H^4. Here $\mathbb{P}\mathrm{er}$ denotes the polarized period space; polarization comes from the restriction of the Fubini-Study form on $\mathbb{C}P^5$.

THEOREM: Let Teich be the Teichmüller space of a 4-dimensional cubic hypersurface, and $\mathsf{Per}: \text{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ its period map. **Then Per is injective, and its image is open.**

Calabi-Yau manifolds

DEFINITION: A compact complex n-manifold M of Kähler type with trivial canonical bundle $K_M := \Lambda^{n,0}(M)$ and $H^{n,0}(M) = \mathbb{C}$ is called **a Calabi-Yau manifold**.

REMARK: A Kähler deformation of a Calabi-Yau manifold is Calabi-Yau (do this as an exercise).

THEOREM: (Bogomolov-Tian-Todorov)
Let $\operatorname{Per}: \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$ be the period map associated with the middle cohomology of a Calabi-Yau manifold. **Then** Teich **admits a structure of a smooth manifold, and** Per **is smooth with non-degenerate differential** (and hence, Per is a local diffeomorphism to its image).

Proof: *F. Catanese, A Superficial Working Guide to Deformations and Moduli, arXiv:1106.1368.* ■

Symmetric spaces

For literature on symmetric spaces, see *A. Besse, Einstein manifolds*, or *P. Petersen, Riemannian Geometry.*

DEFINITION: A symmetric space is a complete Riemannian manifold X such that for all $x \in X$ there exists an isometry ι_x of X fixing x and acting as -1 in $T_x X$.

EXERCISE: Prove that **isometry group acts transitively on any symmetric manifold.** This implies that **any symmetric space has form G/K, where G is a Lie group and K its compact subgroup.**

DEFINITION: A symmetric space is **irreducible** if it is not locally isometric to a product.

PROPOSITION: Let M be a non-compact irreducible symmetric space. **Then $M = G/K$, where G is a simple Lie group and K its maximal compact subgroup**.

Cartan duality

THEOREM: (Cartan duality) For any non-compact irreducible symmetric space G/K, denote by G_1 the compact form of the Lie group G. **Then G_1/K is a compact symmetric space, and any compact symmetric space is obtained this way.**

REMARK: Cartan duality **defines a bijective correspondence between compact and non-compact irreducible symmetric spaces.**

Exercise 1: Let (M, I, g) be an almost complex symmetric space. Assume that the isometry ι_x preserves I. **Prove that (M, I, g) is Kähler.**

REMARK: É. Cartan classified Kähler irreducible symmetric spaces. There are 4 series, associated with classical Lie groups, and 2 special examples, associated with special Lie groups E_6 and E_7.

Two of these examples are associated with variations of Hodge structures.

$\mathrm{Gr}_{++}(V)$ is a Kähler symmetric space

Recall that $O(p,q)$, $p,q>0$ has 4 connected components. **They are distinguished by preserving the orientation on positive and on negative definite subspaces of maximal dimension.** Denote by $O^+(2,n)$ **a subgroup consisting of isometries which preserve orientation on positive 2-planes;** it has 2 connected components.

PROPOSITION: Let V be a vector space of signature $(2,n)$, and $\mathrm{Gr}_{++}(V)$ the Grassmannian of positive oriented planes. **Then $\mathrm{Gr}_{++}(V)$ is a Kähler symmetric space, and $O^+(V)$ acts on $\mathrm{Gr}_{++}(V)$ preserving the complex structure and the Kähler metric.**

Proof. Step 1: It is not hard to see that $T_W \mathrm{Gr}(V) = \mathrm{Hom}(W, V/W)$ where $W \subset V$ is a point of the Grassman space $\mathrm{Gr}(V)$ considered as a subspace in V. In our case, the scalar product is positive definite on W and negative definite on V/W, defining an $O(V)$-invariant positive definite scalar product on $T\,\mathrm{Gr}_{++}(V)$. Consider an isometry $\tau \in O^+(V)$ acting trivially on W and as $-\mathrm{Id}$ on $W^\perp$. Then τ acts on $\mathrm{Gr}_{++}(V)$ preserving W and acting as $-\mathrm{Id}$ on $T_W \mathrm{Gr}(V) = \mathrm{Hom}(W, V/W)$. **Therefore, $\mathrm{Gr}_{++}(V)$ is a symmetric space.**

$\mathrm{Gr}_{++}(V)$ is a Kähler symmetric space (2)

PROPOSITION: Let V be a vector space of signature $(2, n)$, and $\mathrm{Gr}_{++}(V)$ the Grassmannian of positive oriented planes. **Then $\mathrm{Gr}_{++}(V)$ is a Kähler symmetric space, and $O^+(V)$ acts on $\mathrm{Gr}_{++}(V)$ preserving the complex structure and the Kähler metric.**

Proof. Step 1: ...Therefore, $\mathrm{Gr}_{++}(V)$ is a symmetric space.

Step 2: Since W is oriented and positive definite, it is equipped with a counterclockwise rotation by $\pi/2$, denoted by I_W. Extend this map to $\mathrm{Hom}(W, V/W)$ acting as identity on V/W. **This defines an $O^+(V)$-invariant almost complex structure on $O^+(V)$.**

Step 3: As follows from Exercise 1, **these operators define a Kähler structure on $\mathrm{Gr}_{++}(V)$.** ■

EXERCISE: Prove that **an $O^+(V)$-invariant complex structure on $\mathrm{Gr}_{++}(V)$ is unique, and $O^+(V)$-invariant metric is unique up to a scalar multiplier.**

Polarized Hodge structures of K3 type

DEFINITION: A polarized Hodge structure of K3 type is a polarized Hodge structure $V, V_{\mathbb{C}} = V^{2,0} \oplus V^{1,1} \oplus V^{0,2}$ such that $\dim V^{2,0} = 1$.

REMARK: Clearly, the line $V^{2,0} \subset V_{\mathbb{C}}$ uniquely determines a polarized Hodge structure: $V^{0,2} = \overline{V^{2,0}}$, and $V^{1,1} = (V^{2,0} \oplus V^{0,2})^{\perp}$. The Hodge-Riemann condition is equivalent to $h|_{V^{2,0}}$ being Hermitian and $h(V^{2,0}, V^{2,0}) = 0$. This gives the following claim

CLAIM: The period space of polarized Hodge structures of K3 type is

$$\operatorname{Per} := \{l \in \mathbb{P}V \quad | \quad h(l,l) = 0, h(l,\bar{l}) > 0.\}$$

■

Period space for Hodge structures of K3 type

PROPOSITION: Let V be a real vector space, and h a symmetric form of signature $2, n$. The period space

$$\operatorname{Per}\{l \in \mathbb{P}V \;\;|\;\; h(l,l) = 0, h(l,\bar{l}) > 0.\}$$

is identified with $SO(2,n)/SO(2) \times SO(n)$, which is a Grassmannian of positive oriented 2-planes in V.

Proof. Step 1: Given $l \in \mathbb{P}H^2(M,\mathbb{C})$, **the space generated by $\operatorname{Im} l, \operatorname{Re} l$ is 2-dimensional,** because $h(l,l) = 0, h(l,\bar{l}) > 0$ implies that $l \cap V = 0$.

Step 2: This 2-dimensional plane is positive, because $h(\operatorname{Re} l, \operatorname{Re} l) = h(l+\bar{l}, l+\bar{l}) = 2h(l,\bar{l}) > 0$.

Step 3: Conversely, for any positive 2-dimensional plane $W \subset V$, **the quadric $\{l \in W \otimes_{\mathbb{R}} \mathbb{C} \;\;|\;\; h(l,l) = 0\}$ consists of two lines;** a choice of a line is determined by orientation. ■

REMARK: We have shown that **the period space of polarized Hodge structures of K3 type is a Kähler symmetric space.**