Variations of Hodge structures

lecture 3: Gauss-Manin local systems

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Locally constant sheaves

DEFINITION: Let $\mathcal{F}$ be a sheaf on M which takes a connected non-empty open subset $U \subset M$ to a vector space or abelian group $\mathbb{V}$. Extend $\mathcal{F}$ to all open sets using the gluing axiom. Then $\mathcal{F}$ is called **the constant sheaf**, denoted $\mathbb{V}_M$.

EXERCISE: Prove that **the constant sheaf $\mathbb{V}_M$ exists, and is unique up to isomorphism.**

EXERCISE: Let W be an open set in M, and S_W its set of connected components. Prove that $\mathbb{V}_M(W) = \mathbb{V}^{|S_W|}$.

DEFINITION: A **locally constant sheaf** is a sheaf which is locally isomorphic to a constant sheaf.

Local systems

EXAMPLE: Let $\pi: M' \longrightarrow M$ be a covering. Given $U \subset M$, let S_U be the set of connected components of $\pi^{-1}(U)$, and set $\mathcal{F}(U) = \mathbb{V}^{|S_W|}$. We are going to define the restriction map r as follows. For an open subset $W \subset U$, consider the map $S_W \longrightarrow S_U$ induced by the natural embedding $\pi^{-1}(W) \stackrel{j}{\hookrightarrow} \pi^{-1}(U)$. For each direct sum component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \in \operatorname{im} j$, let $r_u: \mathbb{V}_u \longrightarrow \mathbb{V}_{j(u)}$ be identity. For a component $\mathbb{V}_u \subset \mathbb{V}^{|S_U|}$ corresponding to $u \notin \operatorname{im} j$, we set $r_u = 0$. Then $r := \bigoplus_{u \in S_U} r_u: \bigoplus_{u \in S_U} \mathbb{V} \longrightarrow \bigoplus_{w \in S_W} \mathbb{V}$. **This defines a locally constant sheaf on M (prove it).**

DEFINITION: A local system is a locally constant sheaf of vector spaces or abelian groups.

REMARK: A local system on M **is uniquely determined by a representation of the fundamental group $\pi_1(M)$.** In particular, **any local system on a simply connected space is trivial (prove it).**

DEFINITION: A connection on a vector bundle is **flat** if its curvature vanishes.

Riemann-Hilbert correspondence

THEOREM: The category of locally constant sheaves of vector spaces **is naturally equivalent to the category of vector bundles on M equipped with a flat connection.**

Proof. Step 1: Consider a constant sheaf $\mathbb{R}_M$ on M. This is a sheaf of rings, and any locally constant sheaf is a sheaf of $\mathbb{R}_M$-modules.

Let $\mathbb{V}$ be a locally constant sheaf, and $B := \mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$. Since $\mathbb{V}$ is locally constant, the sheaf B is a locally free sheaf of C^∞-modules, that is, a vector bundle. Let $U \subset M$ be an open set such that $\mathbb{V}|_U$ is constant. If $v_1, ..., v_n$ is a basis in $\mathbb{V}(U)$, all sections of $B(U)$ have a form $\sum_{i=1}^n f_i v_i$, where $f_i \in C^\infty U$. Define the connection ∇ by $\nabla\left(\sum_{i=1}^n f_i v_i\right) = \sum df_i \otimes v_i$. This connection is flat because $d^2 = 0$. It is independent from the choice of v_i because v_i is defined canonically up to a matrix with constant coefficients. **We have constructed a functor from locally constant sheaves to flat vector bundles**.

Step 2: Let now (B, ∇) be a flat bundle over M. The corresponding object $\mathbb{B}(M)$ in the category of locally constant sheaves takes $U \subset M$ and maps it to the space $\mathbb{B}(U)$ of parallel sections of B over U. For any simply connected U, and any $x \in U$, the space $\mathbb{B}(U)$ is identified with a vector space $B|_x$, hence $\mathbb{B}(U)$ is locally constant. Clearly, $B = \mathbb{B} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M$, hence **this construction gives an inverse functor to $\mathbb{V} \mapsto (\mathbb{V} \otimes_{\mathbb{R}_M} \mathbb{C}^\infty M, \nabla)$.** ■

Smooth submersions

DEFINITION: Let $\pi : M \longrightarrow M'$ be a smooth map of manifolds. This map is called **submersion** if at each point of M the differential $D\pi$ is surjective, and **immersion** if it is injective.

CLAIM: Let $\pi : M \longrightarrow M'$ be a submersion. Then each $m \in M$ has a neighbourhood $U \cong V \times W$, where V, W are smooth and $\pi|_U$ **is a projection of $V \times W = U \subset M$ to $W \subset M'$ along V.**

Proof: Follows from the inverse function theorem. ■

THEOREM: ("Ehresmann's fibration theorem")
Let $\pi : M \longrightarrow M'$ be a smooth submersion of compact manifolds. **Prove that π is a locally trivial fibration.**

Proof: Next slide.

DEFINITION: Vertical tangent space $T_\pi M \subset TM$ of a submersion $\pi : M \longrightarrow M'$ is the kernel of $D\pi$.

Ehresmann connections

DEFINITION: Let $\pi : M \longrightarrow Z$ be a smooth submersion, with $T_\pi M$ **the bundle of vertical tangent vectors** (vectors tangent to the fibers of π). An **Ehresmann connection** on π is a sub-bundle $T_{\text{hor}} M \subset TM$ such that $TM = T_{\text{hor}} M \oplus T_\pi M$. The **parallel transport** along the path $\gamma : [0, a] \longrightarrow Z$ associated with the Ehresmann connection is a diffeomorphism

$$V_t : \pi^{-1}(\gamma(0)) \longrightarrow \pi^{-1}(\gamma(t))$$

obtained by integrating a vector field $A_{\text{hor}} = \frac{dV_t}{dt} \in T_{\text{hor}} M$ which is a preimage of a vector field $\frac{d}{dt}$ on γ.

CLAIM: Let $\pi : M \longrightarrow Z$ be a smooth fibration with compact fibers. Then **the parallel transport, associated with the Ehresmann connection, always exists.**

Proof: Follows from existence and uniqueness of solutions of ODEs. ■

The Gauss-Manin bundle (a less formal version)

DEFINITION: (a less formal version)
Let $\pi : M \longrightarrow X$ be a proper, smooth submersion. Denote by $\Lambda^k_\pi(M)$ the infinite-dimensional **bundle of fiberwise differential forms** on M. The sections of this bundle are families of differential forms on each fiber $\pi^{-1}(x)$, smoothly depending on $x \in X$. **A fiberwise de Rham differential** $d_\pi : \Lambda^k_\pi(M) \longrightarrow \Lambda^{k+1}_\pi(M)$. takes a fiberwise form to its fiberwise differential. **Its cohomology is a finite-dimensional vector bundle of families of cohomology classes on on each fiber $\pi^{-1}(x)$, smoothly depending on $x \in X$.** Since π is a locally trivial fibration, this cohomology sheaf is a finite-dimensional vector bundle, called **the Gauss-Manin bundle**.

REMARK: Soon enough, we will equip this bundle with a flat connection, **in such a way that the corresponding local system is the local system of cohomology of fibers of π.**

The Gauss-Manin bundle (a more formal version)

DEFINITION: (a more formal version)
Let $\pi : M \longrightarrow X$ be a proper, smooth submersion. Consider an ideal J in the ring $\Lambda^* M$ generated by $\pi^*\Lambda^1 X$. Clearly, J is a differential ideal. **The quotient algebra $\Lambda^*(M)/J$ is the algebra of fiberwise differential forms on M, and the differential induced on $\Lambda^*(M)/J$ is is the fiberwise de Rham differential**. By construction, the fiberwise de Rham differential is $\pi^* C^\infty X$-linear, hence its cohomology can be interpreted as a sheaf of $C^\infty X$-modules, which is clearly locally free and has finite rank. Locally free sheaves of $C^\infty X$-modules are the same as vector bundles on X; the bundle of cohomology of the fiberwise de Rham differential is called **the Gauss-Manin bundle**.

Gauss-Manin connection

DEFINITION: Let $\pi : M \longrightarrow X$ be a proper, smooth submersion, with fiber F and connected base. Consider the bundle $R^i\pi_*(\mathbb{C}_M)$ with fiber $H^i(F_x)$ at each point $x \in X$; we consider $R^i\pi_*(\mathbb{C}_M)$ as the bundle of fiberwise closed form up to fiberwise exact. Fix an Ehresmann connection on π, For any vector field $A \in TX$, denote by A_{hor} its **horizontal lift**, that is, a horizontal field $A_{\text{hor}} \in T_{\text{hor}}M$ which is projected to A. Restricted to any path $\gamma \in X$, the fibration π is a product, and $\operatorname{Lie}_{A_{\text{hor}}}$ maps a fiberwise closed form η to a fiberwise closed form. **The Gauss-Manin connection** $\nabla_A[\eta]$ takes $[\eta] \in R^i\pi_*(\mathbb{C}_M)$ to the cohomology class of $[\operatorname{Lie}_{A_{\text{hor}}}\eta] \in R^i\pi_*(\mathbb{C}_M)$.

REMARK: This definition can be made more formal if we realize that $\operatorname{Lie}_{A_{\text{hor}}} J \subset J$, where $J \subset \Lambda^*M$ is the ideal generated by $\pi^*\Lambda^1X$, and indentify the fiberwise differential forms with Λ^*M/J. **The Gauss-Manin connection is the action of $\operatorname{Lie}_{A_{\text{hor}}}$ on Λ^*M/J.**

Gauss-Manin connection: basic properties

CLAIM: The Gauss-Manin connection is independent from the choice of Ehresmann connection.

Proof: Let A_{hor} and A'_{hor} be horizontal lifts of A for two different Ehresmann connections. Then $\operatorname{Lie}_{A_{\text{hor}}} - \operatorname{Lie}_{A'_{\text{hor}}}$ is a Lie derivative along a vector field tangent to a fiber. **By Cartan formula,** $\operatorname{Lie}_v \eta = di_v + i_v d$ **maps closed forms to exact.** ■

CLAIM: The Gauss-Manin connection is flat.

Proof: By Ehresmann theorem, locally in X the fibration π is trivial. This trivialization **defines an Ehresmann connection for which the horizontal lifts of commuting vector fields commute.** ■.

Gauss-Manin connection and the Hodge decomposition

THEOREM: (Griffiths transversality condition)
Let $\pi: M \longrightarrow X$ be a proper, holomorphic submersion with Kähler fibers, $B := R^d\pi_*(M)$ the bundle of cohomology, and $\nabla: B \longrightarrow B \otimes \Lambda^1 X$ the Gauss-Manin connection. Consider the fiberwise Hodge decomposition, $B = \bigoplus_{p+q=d} B^{p,q}$, and let $\theta^{1,0}$, $\theta^{0,1} \in T_{\mathbb{C}}X$ be vector fields of type $(1,0)$ and $(0,1)$. **Then** $\nabla_{\theta^{1,0}}(B^{p,q}) \subset B^{p,q} \oplus B^{p-1,q+1}$ **and** $\nabla_{\theta^{0,1}}(B^{p,q}) \subset B^{p,q} \oplus B^{p+1,q-1}$.

Proof: Let η be a fiberwise closed (p,q)-form. Since π is holomorphic, it preserves the types, hence we can assume that the horisontal lifts $\theta^{1,0}_{\text{hor}}$, $\theta^{0,1}_{\text{hor}}$ has the same Hodge types as $\theta^{1,0}$, $\theta^{0,1}$. By Cartan formula, $\nabla_{\theta^{1,0}}[\eta]$ is the cohomology class of the restriction of $i_{\theta^{1,0}}d\eta + di_{\theta^{1,0}}\eta$ to the fibers of π. **The first term is a form of type $(p,q)+(p-1,q+1)$, because it is obtained from contraction of a $(p+1,q)+(p,q+1)$-form with $(1,0)$-vector field, and the second term is a differential of a form $i_{\theta^{1,0}}\eta$ of Hodge type $(p-1,q)$, hence it has Hodge type $(p,q)+(p-1,q+1)$.** ■

COROLLARY: Consider a term of the Hodge filtration $F_{d+1} := B^{p-r,q} \oplus B^{p-r-1,q+1} \oplus ... \oplus B^{p-r-d,q+d}$. **Then** $\nabla_{\theta^{0,1}}F_{d+1} \subset F_{d+1}$. ■

$\overline{\partial}$-operator on vector bundles

DEFINITION: Holomorphic vector bundle on a complex manifold M is a locally trivial sheaf of $\mathcal{O}_M$-modules. The sheaf $B \otimes_{\mathcal{O}_M} C^\infty M$ is a locally free sheaf of $C^\infty M$-modules, that is, a smooth vector bundle. It is called **the $C^\infty M$-vector bundle underlying B.**

DEFINITION: Let B be a holomorphic vector bundle on M. Consider an operator $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where b is a holomorphic section of B, and f smooth. This operator is called **a holomorphic structure operator** on B. **It is well-defined because $\overline{\partial}$ is $\mathcal{O}_M$-linear**, and $B_{C^\infty} = B \otimes_{\mathcal{O}_M} C^\infty M$.

REMARK: The kernel of $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ **coincides with the image of B** under the natural sheaf embedding $B \hookrightarrow B_{C^\infty}$, with $b \longrightarrow b \otimes 1$.

DEFINITION: A **$\overline{\partial}$-operator** on a smooth complex vector bundle V over a complex manifold is a differential operator $V \stackrel{\overline{\partial}}{\longrightarrow} \Lambda^{0,1}(M) \otimes V$ satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for any $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator **can be extended to**

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using the Leibnitz identity $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, for all $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

Koszul-Malgrange theorem

REMARK: For any holomorphic bundle, one has $\overline{\partial}^2 = 0$. Indeed, a holomorphic bundle admits a local trivialization.

THEOREM: (Kozsul-Malgrange)
Let $\overline{\partial}: \ V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator on a complex vector bundle, satisfying $\overline{\partial}^2 = 0$, where $\overline{\partial}$ is extended to

$$V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M) \otimes V \xrightarrow{\overline{\partial}} \Lambda^{0,3}(M) \otimes V \xrightarrow{\overline{\partial}} ...$$

as above. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic bundle of the same rank, and $V = B_{\mathbb{C}^\infty}$.**

Proof: The proof uses the same argument as used to prove the Newlander-Nirenberg theorem. ■

COROLLARY: A holomorphic vector bundle **is the same as a smooth vector bundle equipped with a holomorphic structure operator satisfying $\overline{\partial}^2 = 0$.** ■

Holomorphic structure operator

REMARK: Further on, we shall use the same letter for the holomorphic vector bundle and for the underlying smooth bundle. When we need to specify the holomorphic structure $\overline{\partial}$, we say "holomorphic bundle $(B,\overline{\partial})$".

DEFINITION: Let V be a smooth complex vector bundle with connection $\nabla : V \longrightarrow \Lambda^1(M) \otimes V$ and holomorphic structure $\overline{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$. Consider the Hodge type decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$, where

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that **the connection ∇ is compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

CLAIM: Let (B,∇) be a vector bundle with connection, compatible with the holomorphic structure $\overline{\partial}$, and $B_0 \subset B$ a sub-bundle. **Then B_0 is underlies a holomorphic sub-bundle of $(B,\overline{\partial})$ if and only if $\overline{\partial}(B_0) \subset B_0 \otimes \Lambda^{0,1}(M)$.** ■

Hodge structures (reminder)

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb C} = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb Q}$ such that $V_{\mathbb R} = V_{\mathbb Q} \otimes \mathbb R$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb Z} \subset V_{\mathbb Q}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

REMARK: Rational structure on a real vector space V is a $\mathbb Q$-subspace $V_{\mathbb Q} \subset V$ such that $V = V_{\mathbb Q} \otimes_{\mathbb Q} \mathbb R$. **Integer structure** on a real vector space V is a $\mathbb Z$-sublattice $V_{\mathbb Z} \subset V$ such that $V = V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$.

REMARK: A real Hodge structure $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ on $V_{\mathbb R}$ is rational (integer) **if $V_{\mathbb R}$ is equipped with a rational (integer) structure.**

Real structures

DEFINITION: A real structure on a complex vector space V is **an anti-complex involution,** that is, a map $\tau: V \longrightarrow V$ such that $\tau^2 = \operatorname{Id}_V$ and $\tau(\lambda v) = \overline{\lambda} v$.

CLAIM: Let τ be a real structure on a complex vector space V, and V^τ be the space of all τ-invariant vectors. **Then $\dim_{\mathbb{R}} V^\tau = \dim_{\mathbb{C}} V$.** Moreover, **the map $V^\tau \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow V$ taking $x \otimes \lambda$ to λx is an isomorphism.**

Proof: An exercise. ∎

REMARK: The category of real vector spaces is equivalent to the category of complex vector spaces equipped with a real structure, with the functor taking a real vector space $V_{\mathbb{R}}$ to its complexification $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and the inverse taking (V, τ) to V^τ.

Variations of Hodge structures

DEFINITION: Let M be a complex manifold. A **variation of Hodge structures (VHS)** on M is a complex vector bundle (B, ∇) with a flat connection equipped with a parallel anti-complex involution and a Hodge structure, $B = \bigoplus_{p+q=w} B^{p,q}$ which satisfy **"Griffiths transversality condition":** $\nabla^{1,0}(B^{p,q}) \subset B^{p,q} \oplus B^{p+1,q-1}$

DEFINITION: A **polarized VHS** (integer, rational VHS) is a VHS (B, ∇), $B = \bigoplus_{p+q=w} B^{p,q}$ such that ∇ preserves the polarization and the integer or rational lattice.

EXAMPLE: Let $\pi : M \longrightarrow X$ be a proper holomorphic surjective submersion. Consider the bundle $V := R^k\pi_*(\mathbb{C}_M)$ with the fiber in x the k-th cohomology of $\pi^{-1}(x)$, the Hodge decomposition coming from the complex structure on $\pi^{-1}(x)$, and the Gauss-Manin connection. **This defines a variation of Hodge structures,** as we have shown today.

REMARK: Consider a term

$$F_{d+1} := B^{p-r,q} \oplus B^{p-r-1,q+1} \oplus ... \oplus B^{p-r-d,q+d}$$

of the Hodge filtration. **Then $F_{d+1} \subset B$ is a holomorphic sub-bundle.**

Proof: As we have already seen, $\nabla_{\theta^{0,1}} F_{d+1} \subset F_{d+1}$. ∎

Period space (reminder)

DEFINITION: Let V be a real vector space, and $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure. Assume that $p, q \geqslant r$ **The Hodge filtration** is the following filtration on the vector space $V_{\mathbb{C}}$:

$$0 \subset V^{r,w-r} \subset V^{r,w-r} \oplus V^{r+1,w-r-1} \subset V^{r,w-r} \oplus V^{r+1,w-r-1} \oplus V^{r+2,w-r-2} \oplus ...$$

Denote by F_n the n-th term of this filtration, $F_n := \bigoplus_{i=0}^{n-1} V^{r+i,w-r-i}$. Clearly, $V^{p,w-p} = F_{p-r+1} \cap \overline{F}_{w-p-r+1}$ **(prove this)**, hence **the Hodge filtration determines the Hodge structure uniquely.**

REMARK: Two subspaces $W_1, W_2 \subset V$ intersect transversally when $W_1 + W_2 = V$. **Therefore, F_{p-r+1} and $\overline{F}_{w-p-r+1}$ intersect transversally.**

DEFINITION: Let $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure on V. Fix dimensions of all $V^{p,q}$; this determines the dimensions of F_i. **The period space** is the space of all flags $0 \subset F_1 \subset ... \subset F_{w-2r} = V$ such that F_{p-r+1} and $\overline{F}_{w-p-r+1}$ intersects transversally for all p.

CLAIM: The points in the period space **are in bijective correspondence with the set of all Hodge structures on V** having the same numbers $\dim V^{p,w-p}$. ■

REMARK: The period space **is an open subset in the corresponding partial flag space**, which is considered as a complex projective manifold.

Period map

DEFINITION: Let M be a simply connected complex manifold, and $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ a variation of Hodge structures. Since ∇ is flat, the corresponding local system is trivial, and parallel transport identifies all fibers of B and trivializes B. The **period map** is a map taking $m \in M$ to the corresponding point $0 \subset F_1|_m \subset ... \subset F_{w-2r}|_m = B|_m$ in the period space $\mathbb{P}\mathrm{er}$.

CLAIM: The period map $\mathrm{Per} : M \longrightarrow \mathbb{P}\mathrm{er}$ is holomorphic.
Proof: The Hodge filtration is holomorphic, hence **the map which associates to a point $m \in M$ a subspace $F_i|_m \subset B|_m$ is also holomorphic.** Indeed, locally F_i has a holomorphic basis $f_1, ..., f_k$, and the corresponding Plücker map can be expressed as $m \mapsto f_1 \wedge ... \wedge f_k$, where $f_1 \wedge ... \wedge f_k$ is considered as an element of $\mathbb{P}\Lambda^k B = M \times \mathbb{P}\Lambda^k B|_m$. ■

DEFINITION: Let $\pi : \mathcal{X} \longrightarrow X$ be a proper holomorphic submersion with Kähler fibers, and $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ the variation of Hodge structures associated with $B = R^w\pi_*(\mathbb{C}_{\mathcal{X}})$, where $R^w\pi_*(\mathbb{C}_{\mathcal{X}})$ is the flat vector bundle with fiber $H^d(\pi^{-1}(x))$ in $x \in X$. Assume that $\pi_1(X) = 0$. The corresponding **period map** takes $x \in X$ to the point in the period space associated with the Hodge structure $H^d(\pi^{-1}(X)) = \bigoplus_{p+q=w} H^{p,q}(\pi^{-1}(X))$.

COROLLARY: In these assumptions, **the period map $\mathrm{Per} : X \longrightarrow \mathbb{P}\mathrm{er}$ is holomorphic.** ■

Noether-Lefschetz loci

DEFINITION: Let M be a simply connected complex manifold, and $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ a variation of Hodge structures. Assume that $\pi_1(M) = 0$, Since ∇ is flat, the corresponding local system is trivial, and parallel transport identifies all fibers of B and trivializes B. Denote by $B_{\mathbb{R}} \subset B$ the set of fixed points of the anticomplex involution on B. Fix a subspace $V \subset B_{\mathbb{R}}|_x$. **Noether-Lefschetz locus** associated with V is the set of all $x \in X$ such that $V \subset B^{u,u}|_x$, where $u = w/2$.

THEOREM: The Noether-Lefschetz locus **is a complex subvariety of M.**

Proof. Step 1: Let $F_{\text{middle}} := \bigoplus_{p \geqslant q}^{p,q}$. Clearly, F_{middle} is a component of the Hodge filtration. Since V is real, and $F_{\text{middle}} \cap \overline{F}_{\text{middle}} = B^{u,u}$, **the Noether-Lefschetz locus is the set of all $x \in M$ such that $V \subset F_{\text{middle}}|_x$.**

Step 2: Let $f_1, ..., f_k$ be a holomorphic basis in F_{middle}. Denote by $f \in \Lambda^k B$ the vector $f_1 \wedge ... \wedge f_k$. For each $v \in V$, the set N_v of all $x \in M$ such that $v \in F_{\text{middle}}|_x$ **is the zero set of a holomorphic section $v \wedge f \in \Lambda^{k+1} B$,** hence it is a complex subvariety in M. Now, the Noether-Lefschetz locus is $\bigcup_{v \in V} N_v$.

■