Variations of Hodge structures

lecture 4: Deligne-Griffiths-Schmid's fixed part theorem

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Hodge structures (reminder)

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb C} = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb Q}$ such that $V_{\mathbb R} = V_{\mathbb Q} \otimes \mathbb R$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb Z} \subset V_{\mathbb Q}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

REMARK: **Rational structure** on a real vector space V is a $\mathbb Q$-subspace $V_{\mathbb Q} \subset V$ such that $V = V_{\mathbb Q} \otimes_{\mathbb Q} \mathbb R$. **Integer structure** on a real vector space V is a $\mathbb Z$-sublattice $V_{\mathbb Z} \subset V$ such that $V = V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$.

REMARK: A real Hodge structure $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ on $V_{\mathbb R}$ is rational (integer) **if $V_{\mathbb R}$ is equipped with a rational (integer) structure.**

Variations of Hodge structures (reminder)

DEFINITION: Let M be a complex manifold. A **variation of Hodge structures (VHS)** on M is a complex vector bundle (B, ∇) with a flat connection equipped with a parallel anti-complex involution and a Hodge structure, $B = \bigoplus_{p+q=w} B^{p,q}$ which satisfy **"Griffiths transversality condition":** $\nabla^{1,0}(B^{p,q}) \subset B^{p,q} \oplus B^{p+1,q-1}$.

DEFINITION: A **polarized VHS** (integer, rational VHS) is a VHS (B, ∇), $B = \bigoplus_{p+q=w} B^{p,q}$ such that ∇ preserves the polarization and the integer or rational lattice.

EXAMPLE: Let $\pi: M \longrightarrow X$ be a proper holomorphic surjective submersion. Consider the bundle $V := R^k\pi_*(\mathbb{C}_M)$ with the fiber in x the k-th cohomology of $\pi^{-1}(x)$, the Hodge decomposition coming from the complex structure on $\pi^{-1}(x)$, and the Gauss-Manin connection. **This defines a variation of Hodge structures.**

REMARK: Consider a term

$$F_{d+1} := B^{p-r,q} \oplus B^{p-r-1,q+1} \oplus ... \oplus B^{p-r-d,q+d}$$

of the Hodge filtration. **Then $F_{d+1} \subset B$ is a holomorphic sub-bundle.**

Proof: As we have already seen, $\nabla_{\theta^{0,1}} F_{d+1} \subset F_{d+1}$. ■

Fixed part theorem

All VHS today are assumed to be real and polarized. **We don't assume that the VHS or the polarization are rational.**

The main result today:

THEOREM: (Deligne-Griffiths-Schmid's fixed part theorem) Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base, and b a parallel section of B. **Then all (p,q)-components of b are also parallel.**

Proof: Later.

COROLLARY: Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base. Assume that the monodromy of ∇ is trivial. **Then the Hodge decomposition is preserved by ∇**, that is, **the corresponding variation of Hodge structures is constant.**

Proof: Consider a basis $e_1, ..., e_n$ of $B|_x$ such that each e_i belongs to some $B^{p,q}$. Since the monodromy of ∇ is trivial, we can extend each e_i to a parallel section $\tilde{e}_i$ of B. The Hodge components of each of $\tilde{e}_i$ are parallel, but it has only one Hodge component at x. **Therefore, $\tilde{e}_i$ has only one Hodge component at each point of M**, and the corresponding Hodge decomposition is also constant. ■

Kodaira-Spencer map

Let $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ be a connection on a vector bundle $B = \bigoplus B_i$. We decompose the connection onto two components, $\nabla = \nabla_0 + K$, where ∇_0 takes each component B_i to $B_i \otimes \Lambda^1 M$, and $K(B_i) \subset \bigoplus_{j \neq i} B_j \otimes \Lambda^1 M$.

CLAIM: In this situation, ∇_0 is a connection, and K is $C^\infty M$-linear.
Proof. Step 1: Denote by $\Pi_i: B \longrightarrow B_i$ the projection to component B_i. Denote the projection $\Pi_i: B \otimes \Lambda^1 M \longrightarrow B_i \otimes \Lambda^1 M$ by the same letter. Then $\nabla_0 = \sigma_i \Pi_i(\nabla(\Pi_i(b)))$, hence for each section $b \in B_i$, and $f \in C^\infty M$, one has

$$\nabla_0(fb) = \Pi_i(\nabla(fb)) = \Pi_i(df \otimes b + f\nabla(b)) = df \otimes b + f\nabla_0(b).$$

This is why ∇_0 is a connection.
Step 2: The difference $\nabla - \nabla_0$ applied to $b \in B_i$ is equal to $\sum_{j \neq i} \Pi_j(\nabla(fb))$, and $\Pi_j(\nabla(fb)) = \Pi_j(df \otimes b + f\nabla(b)) = f\Pi_j(\nabla(b))$. **Therefore, K is $C^\infty M$-linear.** ■

REMARK: Let $(B, \nabla, B = \bigoplus B^{p,q})$ be a variation of Hodge structures, and $\nabla = \nabla_0 + K$ its decomposition obtained as above. **The Griffiths transversality condition is rewritten as $K(B^{p,q}) \subset \Lambda^{0,1} \otimes B^{p+1,q-1} \oplus \Lambda^{1,0} \otimes B^{p-1,q+1}$.**

DEFINITION: In these assumptions, we denote the component of K which takes $B^{p,q}$ to $\Lambda^{0,1} \otimes B^{p+1,q-1}$ (for all indices (p,q)) by KS. It is called **the Kodaira-Spencer map.**

Connections and holomorphic structures

DEFINITION: Let V be a smooth complex vector bundle with connection $\nabla : V \longrightarrow \Lambda^1(M) \otimes V$ and holomorphic structure $\overline{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$. Consider the Hodge type decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$, where

$$\nabla^{0,1} : V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0} : V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that **the connection ∇ is compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: A holomorphic Hermitian vector buncle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: Chern connection on a holomorphic Hermitian vector bundle is a unitary connection compatible with the holomorphic structure.

Chern connection: existence and uniqueness

THEOREM: Every holomorphic Hermitian vector bundle **admits a Chern connection, which is unique.**

Proof. Step 1: Given a complex vector bundle B, define **complex conjugate bundle** $\overline{B}$ as the same $\mathbb{R}$-bundle with complex conjugate $\mathbb{C}$-action. Then **a connection ∇ on B defines a connection $\overline{\nabla}$ on $\overline{B}$, with $\overline{\nabla}^{1,0} = \overline{\nabla^{0,1}}$ and $\overline{\nabla}^{0,1} = \overline{\nabla^{1,0}}$.**

Step 2: Define **$\nabla^{1,0}$-operator** on a complex vector bundle B as a map $B \xrightarrow{\nabla^{1,0}} \Lambda^{1,0}(M) \otimes B$, satisfying $\Lambda^{1,0}(fb) = \partial(f) \otimes b + f\nabla^{1,0}(b)$ for any $f \in C^\infty M, b \in B$. **A $\overline{\partial}$-operator on B defines an $\nabla^{1,0}$-operator on $\overline{B}$, and vice versa.**

Step 3: Hermitian form defines an isomorphism of complex vector bundles $B \xrightarrow{g} \overline{B}^*$. Holomorphic structure on B defines a $\overline{\partial}$-operator on $\overline{B} = B^*$, which is the same as $\nabla^{1,0}$-operator $\nabla_g^{1,0}$ on B. **This gives a connection operator $\nabla := \overline{\partial} + \nabla_g^{1,0}$ on B,** which is Hermitian by construction. ∎

REMARK: When people say "the curvature of a holomorphic Hermitian line bundle", **they speak about the curvature of the Chern connection.**

Curvature of the Chern connection

PROPOSITION: Curvaure Θ_B of a Chern connection on B is a (1,1)-form: $\Theta_B \in \Lambda^{1,1}(M) \otimes \operatorname{End}(B)$.

Proof. Step 1: Let B be a Hermitian bundle. Consider the operator $\varphi \xrightarrow{\iota} -\varphi^*$ acting on $\operatorname{End}(B)$, where $\varphi \longrightarrow \varphi^*$ denotes the Hermitian conjugation. Since $\iota^2 = \operatorname{Id}$, and this is an anticomplex operator, it defines the real structure, and its fixed point set is $\mathfrak{u}_B$, the Lie algebra of anti-Hermitian matrices.

Step 2: Since the Chern connection preserves the Hermitian structure g, one has $\nabla(g) = 0$, which gives $\nabla^2(g) = 0$. This means that $\Theta_B \in \Lambda^2 M \otimes \mathfrak{u}_B$, **and this for is real with respect to the real structure defined by ι.**

Step 3: The $(0,2)$-part of the curvature vanishes, because $\bar\partial^2 = 0$. The $(2,0)$-part of the curvature vanishes, because $\iota(\Theta_B) = \Theta_B$, and **any real structure on $\operatorname{End}(B)$ exchanges $\Lambda^{2,0}(M) \otimes \operatorname{End}(B)$ and $\Lambda^{0,2}(M) \otimes \operatorname{End}(B)$.**

■

COROLLARY: For the Chern connection $\nabla = \bar\partial + \nabla^{1,0}$ on B, one has $\Theta_B = \{\nabla^{1,0}, \bar\partial\}$.

COROLLARY: The curvature of a holomorphic Hermitian line bundle is a closed (1,1)-form.

Curvature of the Chern connection on a line bundle

REMARK: Let B he a Hermitian holomorphic line bundle, and $b \in \Gamma(B)$ a nowhere vanishing holomorphic section. Then

$$d|b|^2 = (\nabla^{1,0}b, b) + (b, \nabla^{1,0}b) = 2\operatorname{Re}(\nabla^{1,0}b, b),$$

which gives

$$\nabla^{1,0}b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b.$$

We obtain that $\Theta_B(b) = 2\bar\partial\partial \log |b| b$, hence $\Theta_B = -2\partial\bar\partial \log |b|$.

REMARK: The same formula holds for B of any rank. Consider a holomorphic section b of B, and let $(\Theta(b), b)$ be the corresponding 2-form on M. **Then $(\Theta(b), b)(x, y) = \bar\partial\partial |b|^2(x, y)$.**

The associated graded bundle

DEFINITION: Let $F_0 \subset F_1 \subset ... \subset F_n$ be a filtration on a holomorphic vector bundle. The bundle $\bigoplus F_i/F_{i-1}$ is called **the associated graded vector bundle**. It is also holomorphic.

REMARK: Let $(B, \nabla, B = \bigoplus B^{p,q})$ be a variation of Hodge structures. We identify $B^{p,q}$ as the associated graded bundle to the Hodge filtration on B. Since the Hodge filtration is holomorphic, this puts another holomorphic structure on $\bigoplus B^{p,q}$. This second holomorphic structure **is distinct from the holomorphic structure on B induced by $\overline{\partial} := \nabla^{0,1}$.**

Chern connection on the associated graded bundle of VHS

THEOREM: Let $(B, \nabla, B = \bigoplus B^{p,q}, h)$ be a polarized variation of Hodge structures, and $\nabla = \nabla_0 + K$ the corresponding decomposition of ∇, with $\nabla_0(\bigoplus B^{p,q}) \subset B^{p,q} \otimes \Lambda^1 M$. **Then ∇_0 is the Chern connection $\bigoplus B^{p,q}$, associated with the holomorphic structure on the associated graded bundle defined above, and the Hermitian structure obtained from the polarization.**

Proof. Step 1: To simplify notations, I would assume that the decomposition $B = \bigoplus B^{p,q}$ starts from $(p+q, 0)$. A holomorphic section α of $B^{p,q}$ comes from a holomorphic section $\tilde{\alpha}$ of the corresponding component of the Hodge filtration $F_p = \bigoplus_{i=0}^{i=q} B^{p-i+q,i}$. By Griffiths transversality, $\nabla_0^{0,1}(\tilde{\alpha})$ is equal to the sum of $\nabla^{0,1}(\tilde{\alpha})$ (which vanishes because $\tilde{\alpha}$ is holomorphic) and $K^{0,1}(\tilde{\alpha}) \in F_{p-1} \otimes \Lambda^{0,1}(M)$. The second term vanishes after passing to the associated graded bundle $\bigoplus F_i/F_{i-1}$. **We have shown that $\nabla_0^{0,1}$ is the holomorphic structure operator of the associated graded bundle $\bigoplus B^{p,q}$.**

Step 2: Consider sections $\alpha \in B^{p,q}$ and $\beta \in B^{q,p}$. **The connection ∇_0 is h-orthogonal if**

$$d(h(\alpha, \beta)) = h(\nabla_0 \alpha, \beta) + h(\alpha, \nabla_0 \beta).$$

Since ∇ is h-orthogonal, this equation would follow from $h(K\alpha, \beta) + h(\alpha, K\beta) = 0$. which is clear from the Hodge type decomposition of $K\alpha$ and $K\beta$. ■

The second fundamental form

DEFINITION: Let $0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_1/B_2 \longrightarrow 0$ be an exact sequence of vector bundles, and $\nabla : B_2 \longrightarrow B_2 \otimes \Lambda^1 M$ a connection on B_1. **The second fundamental form** of ∇ is the map $A := B_1 \longrightarrow (B_2/B_1) \otimes \Lambda^1 M$ obtained as a composition of $\nabla\Big|_{B_1} : B_1 \longrightarrow B_2 \otimes \Lambda^1 M$ with the projection $B_2 \otimes \Lambda^1 M \longrightarrow (B_2/B_1) \otimes \Lambda^1 M$.

REMARK: This map **is clearly $C^\infty(M)$-linear.**

The second fundamental form and the curvature

THEOREM: Let (B, h) be a Hermitian vector bundle and ∇ an orthogonal connection, $B_1 \subset B$ a sub-bundle, and $B_2 := B_1^\perp$ its orthogonal complement. Denote by $A_1 : B_1 \longrightarrow B_2 \otimes \Lambda^1 M$, $A_2 : B_2 \longrightarrow B_1 \otimes \Lambda^1 M$ the second fundamental forms, and let ∇_1, ∇_2 be connections in B_1, B_2 obtained by applying $\nabla : B_i \longrightarrow B \otimes \Lambda^1 M$ and projecting to $B_i \otimes \Lambda^1 M$. Then

(i) $\nabla = \nabla_1 + \nabla_2 + A_1 + A_2$.

(ii) Consider the second fundamental forms $A_1 \in \Lambda^1 M \otimes \operatorname{Hom}(B_1, B_2)$ and $A_2 \in \Lambda^1 M \otimes \operatorname{Hom}(B_2, B_1)$. **Then for any $x \in TM$, one has $h(A_1(b_1), b_2) = h(b_1, A_2(b_2))$,** in other words, $A_1 = A_2^T$.

(iii) Let $\Lambda^2 M \otimes \operatorname{End}(B) \xrightarrow{\Pi} \Lambda^2 M \otimes \operatorname{End}(B_1)$, be the orthogonal projection and let Θ_{∇_1}, Θ_∇ be the curvature of ∇_1, ∇. **Then $\Theta_{\nabla_1} + A_1 \wedge A_2 = \Pi(\Theta_\nabla)$.**

Proof: (i) is clear, and (iii) follows from (i), because $\Pi(\Theta_\nabla) = \Pi((\nabla_1 + \nabla_2 + A_1 + A_2)^2) = \Pi(\nabla_1^2 + A_1 \wedge A_2)$. The condition (ii) is implied by taking

$$0 = \nabla(h)(b_1, b_2) = d(h(b_1, b_2)) + h(\nabla(b_1), b_2) + h(b_1, \nabla(b_2)) = \\ = h(A_1(b_1), b_2) + h(b_1, A_2(b_2)).$$

■

The curvature of a holomorphic sub-bundle

Let now $B_1 \subset B$ be a holomorphic sub-bundle of a holomorphic Hermitian bundle (B, h). Then $\nabla^{0,1}(B_1) \subset B_1 \otimes \Lambda^{0,1}(M)$, **which gives** $A_1 \in \Lambda^{1,0}(\operatorname{Hom}(B_1, B_2))$**,** where $B_2 = B_1^\perp$. As before, we denote by $\Lambda : \Lambda^2(M) \otimes \operatorname{End}(B) \longrightarrow \operatorname{End}(B)$ the operator adjoint to multiplication by ω.

CLAIM: Let $B_1 \subset B$ be a holomorphic sub-bundle of a holomorphic Hermitian bundle (B, h), $B_2 := B_1^\perp$, and $A_1 \in \Lambda^1 M \otimes \operatorname{Hom}(B_1, B_2)$, $A_2 \in \Lambda^1 M \otimes \operatorname{Hom}(B_2, B_1)$ the corresponding second fundamental forms. **Then** $\sqrt{-1}\operatorname{Tr}\Lambda(A_1 \wedge A_2) \geqslant 0$**, and the inequality is strict wherever** $A_1 \neq 0$**.**

Proof: For each $z \in T_m^{1,0}M$, the operator $A_1(z)$ is anti-Hermitian and adjoint to $A_2(\bar{z})$ because $h(A_1(b_1), b_2) = h(b_1, A_2(b_2))$. Therefore, the operator $A := A_1 \wedge A_2(z, \bar{z})$ is a square of an anti-Hermitian operator. **The square of an anti-Hermitian operator R is a Hermitian operator with negative eigenvalues:** $h(R(b), R(b)) = -h(R^2(b), b) \geqslant 0$. Since $\Lambda(A_1 \wedge A_2) = \sum_i A_1 \wedge A_2(z_i, \bar{z}_i)$ for some orthonormal basis $z_1, ..., z_n \in T_m^{1,0}M$, and each of these terms is either zero or sign-definite of the same sign, this implies $\sqrt{-1}\operatorname{Tr}\Lambda(A_1 \wedge A_2) = \sqrt{-1}\sum_i A_1 \wedge A_2(z_i, \bar{z}_i) \geqslant 0$, with equality at $m \in M$ only when $A_1\big|_{T_mM} = 0$. ■

Plurisubharmonic functions

DEFINITION: A real (1,1)-form ω on a complex manifold is called **positive** if $\omega(x, Ix) \geqslant 0$ for any $x \in TM$.

DEFINITION: A smooth real-valued function ψ on a complex manifold is called **plurisubharmonic (psh)** if $dd^c\psi = -\sqrt{-1}\,\partial\bar\partial\psi$ is a positive (1,1)-form.

EXAMPLE: **$|x|^2$ on $\mathbb{C}^n$ is psh.**

LEMMA 1: Let $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ be a monotonously non-decreasing, convex smooth function, and $f \in C^\infty M$ plurisubharmonic. **Then $f(\psi)$ is also psh.**

Proof: Chain rule gives

$$dd^c(\mu(f)) = \mu'(f(z))^2 dd^c f + \mu''(f(z)) df \wedge d^c f.$$

■

DEFINITION: A holomorphic Hermitian line bundle L is called **positive** if $-\sqrt{-1}\,\Theta$ is positive, where Θ is its curvature form.

CLAIM: Let v be a holomorphic section of a positive holomorphic Hermitian line bundle L. **Then $|v|^2$ is psh.**

Proof: Curvature of L is $\partial\bar\partial \log|v|^2$, and this form is positive, hence $\psi := \log|v|^2$ is psh. Then $|v|^2 = e^\psi$ is psh by Lemma 1. ■

PROPOSITION: Let (M, I) be a compact complex manifold, and φ a psh function on M. **Then $\varphi = const$.**

Proof: This statement immediately follows from the strong maximum principle, http://verbit.ru/IMPA/CV-2023/slides-cv-27.pdf. ■

Kodaira-Spencer map and parallel sections

THEOREM: Let $(B, \nabla, B = \bigoplus B^{p,q}, h)$ be a polarized variation of Hodge structures, and $\nabla = \nabla_0 + K$ the corresponding decomposition of ∇. Consider a parallel section v of B, and let $v = \sum v^{p,q}$ be its Hodge components. Denote by p_0 the smallest p for which $v^{p,q} \neq 0$, and let and $u := v^{p_0,q_0}$ be the corresponding Hodge component. **Then h is ∇_0-holomorphic, that is, $u \in \ker \nabla_0^{0,1}$ and $h(u, \overline{u})$ is psh.**

Proof. Step 1: To prove that u is ∇_0-holomorphic, we note that $0 = \nabla^{0,1}(v) = \nabla_0^{0,1}(v) + K^{0,1}(v)$. Then $0 = \nabla_0^{0,1}(v^{p,q}) + K^{0,1}(v^{p-1,q+1})$ is the (p,q)-component of $\nabla^{0,1}(v)$, giving $\nabla_0^{0,1}(u) = -K^{0,1}(v^{p_0-1,q_0+1})$. **Since p_0 is the smallest possible, $v^{p_0-1,q_0+1} = 0$, hence $\nabla_0^{0,1}(u) = 0$.**

Step 2: From the same argument as above, $dd^c(\log h(u, \overline{u})) = \Theta_{B^{p_0,q_0}}$, where $\Theta_{B^{p_0,q_0}}$ is the curvature of the Chern connection on B^{p_0,q_0}. However, $\Theta_{B^{p_0,q_0}} = \Theta_\nabla + \{K^{1,0}, K^{0,1}\} = \{K^{1,0}, K^{0,1}\}$ because $\nabla_0 = \nabla + K^{1,0} + K^{0,1}$, and the curvature of the Chern connection has type $(1,1)$ (here $\{\cdot,\cdot\}$ denotes the anticommutator). **It remains to show that the form $R := \{K^{1,0}, K^{0,1}\}$ satisfies $\sqrt{-1}\,(R(u), u)(x, Ix) \geqslant 0$ for all $x \in TM$.**

Step 3: Clearly, $K^{0,1}(u) = 0$, which gives $R(u) = K^{0,1}K^{1,0}u$. Then $(R(u), u) = -(K^{1,0}u, K^{1,0}u)$, because $(K^{1,0})^T = -\overline{K^{0,1}}$; this follows from ∇_0 and ∇ preserving polarization. ■

Fixed part theorem

THEOREM: (Deligne-Griffiths-Schmid's fixed part theorem)
Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base, and v a parallel section of B. **Then all (p,q)-components of v are also parallel.**

Proof. Step 1: Let $v^{p,q}$ are Hodge components of v, and $u := v^{p_0,q_0}$ the non-zero component with smallest p. **We are going to prove that $\nabla(u) = 0$.** Replacing v by $v - u$, we arrive in the same situation as described by the fixed part theorem, but now v has a smaller number r of Hodge components. Repeating this argument, we arrive at the situation when $r = 1$ and then the only Hodge component of v is parallel by construction.

Step 2: Let $v^{p,q}$ are Hodge components of v, and $u := v^{p_0,q_0}$ the non-zero component with smallest p. Then $h(u,u)$ is plurisubharmonic, hence constant, because any plurisubharmonic function on a compact manifold is constant. **Since $dd^c h(u,u) = (K^{1,0}u, K^{1,0}u)$, this implies that $K(u) = 0$, and $\nabla(u) = \nabla_0(u)$.**

Step 3: Since $dd^c h(u,u) = 0$, and u is holomorphic, u satisfies $0 = \partial\bar{\partial} h(u,u) = h(\nabla_0^{1,0} u, \nabla_0^{1,0} u)$. This implies $\nabla_0^{0,1} u = \nabla_0^{1,0} u = 0$. **Then $\nabla(u) = \nabla_0(u) = 0$ (Step 2).** By Step 1, this assertion implies the fixed part theorem. ■