Variations of Hodge structures

lecture 5: Deligne's semisimplicity theorem

Misha Verbitsky

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Hodge structures (reminder)

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight w** on a vector space $V_{\mathbb C} = V_{\mathbb R}\otimes_{\mathbb R}\mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb Q}$ such that $V_{\mathbb R} = V_{\mathbb Q}\otimes\mathbb R$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb Z}\subset V_{\mathbb Q}$. A Hodge structure is equipped with $U(1)$-action, with $u\in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

REMARK: **Rational structure** on a real vector space V is a $\mathbb Q$-subspace $V_{\mathbb Q}\subset V$ such that $V = V_{\mathbb Q}\otimes_{\mathbb Q}\mathbb R$. **Integer structure** on a real vector space V is a $\mathbb Z$-sublattice $V_{\mathbb Z}\subset V$ such that $V = V_{\mathbb Z}\otimes_{\mathbb Z}\mathbb R$.

REMARK: A real Hodge structure $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ on $V_{\mathbb R}$ is rational (integer) **if $V_{\mathbb R}$ is equipped with a rational (integer) structure.**

Tensor product of Hodge structures

DEFINITION: As usual, we interpret the Hodge structure as $U(1)$-action. **This defines the Hodge structure on a tensor product of two Hodge structures:** if $W_{\mathbb C} = \bigoplus_{p+q=w} W^{p,q}$ and $V_{\mathbb C} = \bigoplus_{p+q=v} V^{p,q}$, then $W_{\mathbb C} \otimes_{\mathbb C} V_{\mathbb C} = \bigoplus_{p+q=v+w} (V\otimes W)^{p,q}$, where $(V\otimes W)^{p,q} = \bigoplus_{\substack{p_1+p_2=p\\ q_1+q_2=q}} V^{p_1,q_1}\otimes W^{p_2,q_2}$

DEFINITION: Let $(V, V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q})$ be a Hodge structure of weight w. **By convention, the Hodge structure on V^* has weight $-w$,** $V^*_{\mathbb C} = \bigoplus_{p+q=-w} (V^*)^{p,q})$, where $(V^*)^{p,q} = (V^{-p,-q})^*$.

REMARK: Therefore, for any Hodge structure V, **the space $\operatorname{End}(V) = V^*\otimes V$ is a Hodge structure of weight 0.** The endomorphisms of the Hodge structure are the same as elements $\nu \in \operatorname{End}(V)$ (for real Hodge structures), $\nu\in \operatorname{End}(V_{\mathbb Q})$ for rational Hodge structures and $\nu \in \operatorname{End}(V_{\mathbb Z})$ which commute with the Hodge decomposition. **The latter is equivalent to ν being of Hodge type $(0,0)$.**

Variations of Hodge structures (reminder)

DEFINITION: Let M be a complex manifold. A **variation of Hodge structures (VHS)** on M is a complex vector bundle (B, ∇) with a flat connection equipped with a parallel anti-complex involution and a Hodge structure, $B = \bigoplus_{p+q=w} B^{p,q}$ which satisfy **"Griffiths transversality condition":** $\nabla^{1,0}(B^{p,q}) \subset B^{p,q} \oplus B^{p+1,q-1}$.

DEFINITION: A **polarized VHS** (integer, rational VHS) is a VHS (B, ∇), $B = \bigoplus_{p+q=w} B^{p,q}$ such that ∇ preserves the polarization and the integer or rational lattice.

EXAMPLE: Let $\pi: M \longrightarrow X$ be a proper holomorphic surjective submersion. Consider the bundle $V := R^k\pi_*(\mathbb{C}_M)$ with the fiber in x the k-th cohomology of $\pi^{-1}(x)$, the Hodge decomposition coming from the complex structure on $\pi^{-1}(x)$, and the Gauss-Manin connection. **This defines a variation of Hodge structures**.

REMARK: Consider a term

$$F_{d+1} := B^{p-r,q} \oplus B^{p-r-1,q+1} \oplus ... \oplus B^{p-r-d,q+d}$$

of the Hodge filtration. **Then $F_{d+1} \subset B$ is a holomorphic sub-bundle.**

Proof: As we have already seen, $\nabla_{\theta^{0,1}} F_{d+1} \subset F_{d+1}$. ∎

Fixed part theorem (reminder)

The main result today:

THEOREM: (Deligne-Griffiths-Schmid's fixed part theorem)
Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base, and b a parallel section of B. **Then all (p,q)-components of b are also parallel.**

Proof: Lecture 4. ■

REMARK: This statement **also holds when M is quasiprojective (we prove it later in this course, if time permits).**

COROLLARY: Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base. Assume that the monodromy of ∇ is trivial. **Then the Hodge decomposition is preserved by ∇**, that is, **the corresponding variation of Hodge structures is constant.**

Proof: Consider a basis $e_1, ..., e_n$ of $B|_x$ such that each e_i belongs to some $B^{p,q}$. Since the monodromy of ∇ is trivial, we can extend each e_i to a parallel section $\tilde{e}_i$ of B. The Hodge components of each of $\tilde{e}_i$ are parallel, but it has only one Hodge component at x. **Therefore, $\tilde{e}_i$ has only one Hodge component at each point of M,** and the corresponding Hodge decomposition is also constant. ■

Rigidity theorem

REMARK: We proved the fixed part theorem for compact base manifold, but in fact it is true for quasiprojective base. All the results today depend only on the fixed part theorem, and **they remain valid over the quasiprojective base as well.**

COROLLARY: (Rigidity theorem)
Let V, W be variations of polarized Hodge structures over a compact (or quasiprojective) manifold M. Assume that there exists a bundle isomorphism $\varphi: \ V \longrightarrow W$ commuting with the flat connections **(this is equivalent to having an isomorphism of the corresponding local systems).** Assume, moreover, that φ induces an isomorphism of Hodge structures at a point $m \in M$. **Then φ induces an isomorphism of Hodge structures**.

Proof: Consider the bundle $V^* \otimes W$. Since a tensor product of VHS is VHS, $V^* \otimes W$ is a VHS, and φ its parallel section. Then all Hodge components of φ are also parallel. However, at m, the map $\varphi: \ V|_m \longrightarrow W|_m$ preserves the Hodge structure, hence it has type (p,p). **The rest of the Hodge components of φ vanish at m; being parallel, they vanish everywhere.**
■

Deligne's semisimplicity theorem

DEFINITION: We further weaken the notion of real Hodge structures, defining **a complex Hodge structure on a vector space A**, which is a decomposition $A = \bigoplus A^{p,q}$, without the assumption $\overline{A^{p,q}} = A^{q,p}$. **A complex VHS** is a decomposition $V = \bigoplus V^{p,q}$ of a flat vector bundle (V, ∇) such that ∇ acts on $V = \bigoplus V^{p,q}$ satisfying the Griffiths transversality.

THEOREM: (Deligne's semisimplicity theorem)
Let $(V, \nabla, V = \bigoplus V^{p,q})$ be an integer, polarized VHS over a compact (or quasiprojective) base. **Then the flat bundle (V, ∇) can be decomposed as $V = \oplus_i L_i \otimes_{\mathbb{C}} W_i$, where L_i are flat bundles with irreducible monodromy, and W_i complex vector spaces.** Moreover, **each L_i is equipped with a structure of a complex VHS, and each W_i with a complex Hodge structure,** in such a way that this decomposition is compatible with the Hodge structures.

Proof: Later today.

REMARK: This decomposition **is not necessarily compatible with the integer (or even rational) structure** on V.

Monodromy of VHS is semisimple

We start from the following

Proposition 1: In assumptions of Deligne's Semisimplicity theorem, **the representation of $\pi_1(M)$ associated with (V, ∇) is semisimple** (that is, a direct sum of irreducible $\mathbb{C}$-representations).

Proof. Step 1: Flat bundles are the same as representations of the fundamental group, hence every flat bundle contains a subbundle with irreducible monodromy. **Let r the the smallest rank of a non-zero irreducible flat subbundle of V.**

Step 2: Let W be the sum of all irreducible sub-bundles of V of rank r. **To prove Proposition 1, it suffices to show that W is an integer VHS,** that is, $W = W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and W is fixed by the Hodge $U(1)$-action. Indeed, in this case, the orthogonal component $W^\perp$ taken with respect to the polarization is also a VHS, hence we decomposed (V, ∇) onto a direct sum of VHS, with W being by construction semisimple. Semisimplicity follows by induction on $\operatorname{rk} V$.

Monodromy of VHS is semisimple (2)

Step 1-2: L Let r the the smallest rank of a non-zero irreducible flat sub-bundle of V, and W the sum of all irreducible sub-bundles of V of rank r. **To prove Proposition, it suffices to show that W is an integer VHS,** that is, admits an integer lattice and fixed by the Hodge $U(1)$-action.

Step 3: Fix $m \in M$. We denote the fiber $V|_m$ by $\mathbb{V}$, and use $\mathbb{V}_{\mathbb{Q}}$ for its rational lattice. Since $\mathbb{V}$ is a complexification of $\mathbb{V}_{\mathbb{Q}}$, the group $\operatorname{Aut}_{\mathbb{Q}}(\mathbb{C})$ acts on $\mathbb{V}$, also acting on the set of irreducible sub-representations. By definition, a subspace $A \subset \mathbb{V}$ **is defined over $\mathbb{Q}$** if $A = A_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, where $A_{\mathbb{Q}} \subset \mathbb{V}_{\mathbb{Q}}$ is a subspace. Clearly, a subspace $A \subset \mathbb{V}$ is defined over $\mathbb{Q}$ if and only if it is $\operatorname{Aut}_{\mathbb{Q}}(\mathbb{C})$-invariant. **Therefore, $W|_m \subset \mathbb{V}$ is defined over $\mathbb{Q}$.** To see that it is defined over $\mathbb{Z}$, we notice that the monodromy of $V_{\mathbb{Q}}$ is integral, hence the monodromy of any sub-bundle, defined over $\mathbb{Q}$, is also integral.

Step 4: For any $t \in U(1)$, denote by $t(W)$ the result of Hodge rotation applied to W. **We are going to show that $t(W)$ is a flat sub-bundle in V.** Let $d := \operatorname{rk} W$. Since $W = W_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, the monodromy of $\Lambda^d W$ is ± 1. **(*) This is the only point in the proof of Deligne's semisimplicity where we use the integer or the rational structure on (V, ∇)** Then $(\Lambda^d W)^{\otimes 2} \subset (\Lambda^d V)^{\otimes 2}$ is a trivial flat bundle. Fixed Part Theorem implies that $t(\Lambda^d W)^{\otimes 2}$ is flat for any $t \in U(1)$. Then $t(\Lambda^d W) = \Lambda^d tW$ is flat, hence tW is also flat.

Monodromy of VHS is semisimple (3)

Step 1-2: L Let r the the smallest rank of a non-zero irreducible flat sub-bundle of V, and W the sum of all irreducible sub-bundles of V of rank r. **To prove Proposition, it suffices to show that W is a VHS,** that is, fixed by the Hodge $U(1)$-action.

Step 3-4: For any $t \in U(1)$, denote by $t(W)$ the result of Hodge rotation applied to W. **We proved that $t(W)$ is a flat sub-bundle of V.**

Step 5: Let $W_0 \subset W$ be an irreducible sub-bundle of rank r, and $W_1 \subset W$ a complementary sub-bundle such that $W = W_0 \oplus W_1$. Such a complement exists, because W is semisimple. The projection $\Pi_1 : W \longrightarrow W_0$ along W_1 is by construction ∇-parallel, hence all its Hodge components are also parallel. **Then $t\Pi_1 : tW \longrightarrow tW_0$ is parallel as well.**

Step 5: This implies that tW is a direct sum of rank r sub-representations of $\pi_1(M)$. Since W is a sum of all such representations, we obtain that $W = tW$, and therefore $W \subset V$ is a sub-VHS. By Step 2, this implies Proposition 1. ■

VHS on semisimple flat bundles

REMARK: In assumptions of Deligne's Semisimplicity theorem, consider $\mathbb{V} := V|_m$ as a representation of $\pi_1(M,m)$. By Proposition 1, this representation is semisimple. **This gives a decomposition $V = \bigoplus_i L_i \otimes W_i$, where L_i are flat subbundles defining pairwise non-equivalent representations of $\pi_1(M,m)$, and W_i complex vector spaces.** To prove the Semisimplicity theorem, it remains to equip L_i and W_i with the Hodge structures compatible with that on V.

CLAIM: In these assumptions, $\operatorname{End}\left(\bigoplus_i L_i \otimes W_i\right) = \bigoplus_i \operatorname{End}(W_i)$. Moreover, the space of parallel sections of $\operatorname{End}(V)$ is also identified with $\bigoplus_i \operatorname{End}(W_i)$.

Proof: The first statement is Schur's lemma, and the second is implied by the equivalence of categories of local systems, representations of $\pi_1(M)$, and flat bundles. ■

The Hodge decomposition of the projection operators

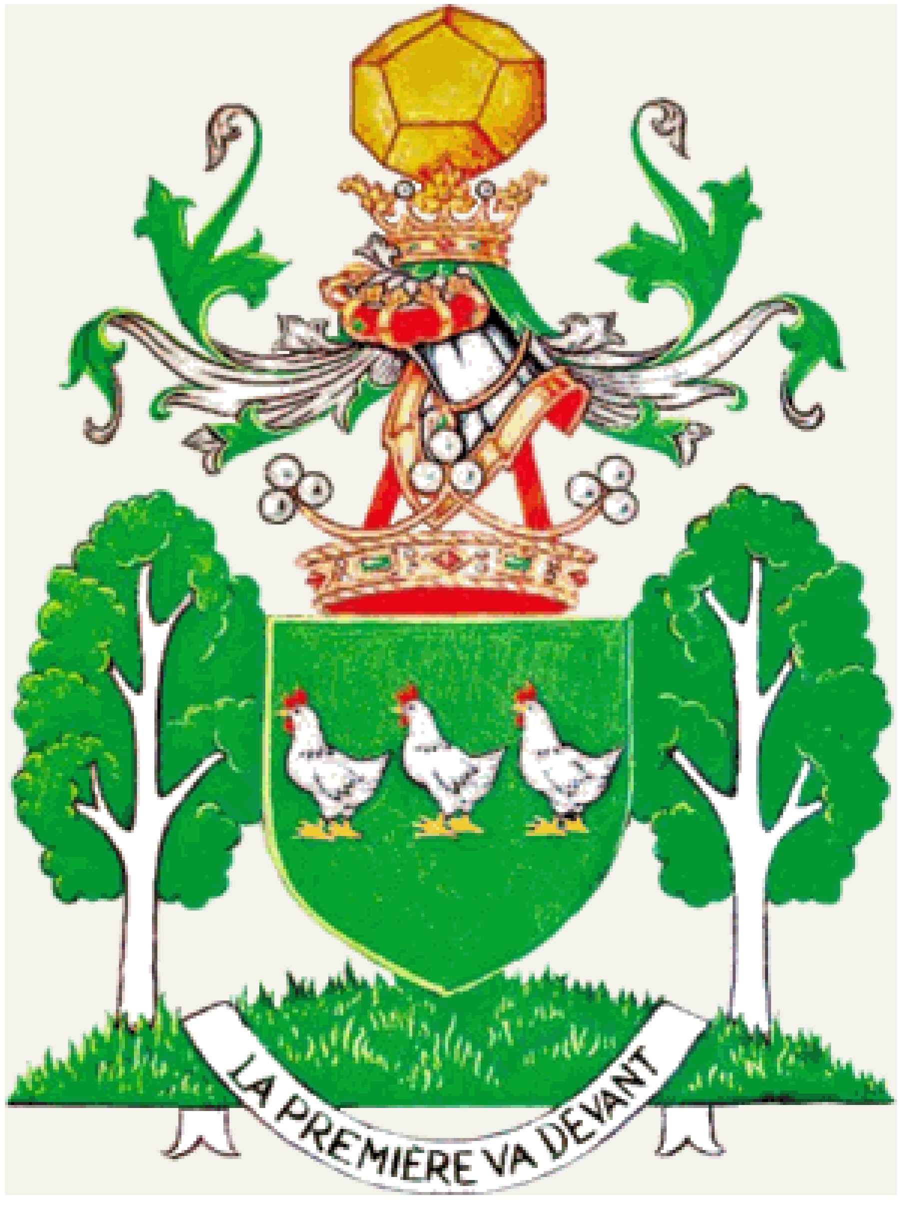
Proposition 2: In assumptions of Deligne's Semisimplicity theorem, consider the decomposition $V = \bigoplus_i L_i \otimes W_i$ constructed above, and let $\operatorname{End}(V) = V \otimes V^*$ be the bundle of endomorphisms equipped with a Hodge structure of weight 0 induced from the tensor product. **Then the projection $\Pi_i : V \longrightarrow L_i \otimes W_i$ has Hodge type (0,0) as a section of $\operatorname{End}(V)$.**

Proof. Step 1: Take any parallel section $\tau \in \operatorname{End}(V)$. By the fixed part theorem, the Hodge components of τ are also parallel. This defines a Hodge structure on the vector space $\operatorname{End}(V)_\nabla$ of parallel sections of $\operatorname{End}(V)$, giving a multiplicative map $U(1) \longrightarrow \bigoplus_i \operatorname{End}(W_i)$. However, for any vector space W_i the natural map $PGL(W_i) \mapsto \operatorname{Aut}(\operatorname{End}(W_i))$ is an isomorphism **(do it as an exercise)**, hence **there exist an $U(1)$-action on each W_i inducing the Hodge structure on $\operatorname{End}(V)_\nabla = \bigoplus_i \operatorname{End}(W_i)$.**

Step 2: The element $\Pi_i \in \bigoplus_i \operatorname{End}(W_i)$ acts as identity on W_i and trivially on the rest of W_j, $j \neq i$, **hence it has Hodge type (0,0).** ■

Remark 1: Consider the Hodge decomposition on W_i constructed in Step 1 above, and let $x \in W_i$ be a vector of pure Hodge type. Denote by p an $U(1)$-invariant projection operator $p: W_i \longrightarrow \langle x \rangle$. **The corresponding section of $\operatorname{End}(V) = \bigoplus_i \operatorname{End}(W_i)$ is of Hodge type (0,0), hence defines a morphism of complex Hodge structures.**

Deligne's s coat of arms



Motto: La première va devant. ("The first one leads.")

Crest: A dodecahedron Or. Supporters: Two trees Vert trunked Argent.

Deligne's s coat of arms (2)

In 2006, Pierre René Deligne was ennobled by Albert II, King of the Belgians. On that occasion, the new Vicomte Deligne designed for himself the above coat of arms, which is inspired by the following nursery rhyme.

https://www.youtube.com/watch?v=vn_nHDUTlbo

Quand trois poules vont aux champs,	*As three hens head for the fields,*
La première va devant,	*The first one leads,*
La deuxième suit la première,	*The second follows the first,*
La troisième est la dernière.	*The third one is last.*
Quand trois poules vont aux champs,	*As three hens head for the fields,*
La première va devant.	*The first one leads.*

The song is a tautology, Deligne explained, "and one can view mathematics as being also (long) chains of tautologies."

From my Master [Alexander Grothendieck], I learned not to take glory in the difficulty of a proof [which may betray a lack of understanding]. The ideal is to be able to paint a landscape in which the proof is obvious." Pierre Deligne (Notices of the AMS, 63, 2, p 250, March 2016)

Deligne's semisimplicity theorem: the proof

THEOREM: (Deligne's semisimplicity theorem)
Let $(V, \nabla, V = \bigoplus V^{p,q})$ be an integer, polarized VHS over a compact (or quasiprojective) base. **Then the flat bundle (V, ∇) can be decomposed as**

$$V = \oplus_i L_i \otimes_{\mathbb{C}} W_i, \quad (*)$$

where L_i are flat bundles with irreducible monodromy, and W_i complex vector spaces. Moreover, **each L_i is equipped with a structure of complex VHS, and each W_i with a complex Hodge structure,** in such a way that the decomposition (*) is compatible with the Hodge structures.

Proof. Step 1: By Proposition 1, $V = \oplus_i L_i \otimes_{\mathbb{C}} W_i$, where L_i are flat bundles with irreducible monodromy, and W_i complex vector spaces. By Proposition 2, each component $L_i \otimes_{\mathbb{C}} W_i$ is $U(1)$-invariant, hence **this decomposition is compatible with the structure of complex VHS.**

Step 2: Consider the operator $p: V \longrightarrow L_i \otimes x$ constructed in Remark 1. Since p is $U(1)$-invariant, its image is a complex VHS; this defines the structure of complex VHS on each L_i. The structure of complex Hodge structure was constructed on each W_i in Step 1 of Proposition 2. By construction, **the $U(1)$-action on $\oplus_i L_i \otimes_{\mathbb{C}} W_i$ is compatible with the $U(1)$-action on W_i and L_i obtained this way.** ■