Variations of Hodge structures

lecture 6: Mumford-Tate group

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Chevalley theorem

DEFINITION: A **tensor representation** of $GL(V)$ is $\bigoplus_i V^{\otimes a_i} \otimes (V^*)^{\otimes b_i}$ for some collection of pairs (a_i, b_i).

DEFINITION: An **algebraic group** is a subgroup of $GL(V)$, defined by a finite collection of polynomial equations on matrix coefficients.

REMARK: It is not very hard to show that **any complex Lie subgroup of $GL(n, \mathbb{C})$ is algebraic;** however, **for $GL(n, \mathbb{R})$ this is false.**

REMARK: Chevalley theorem states that **any algebraic group is completely determined by its projective tensor invariants.**

Chevalley theorem: the proof

THEOREM: Chevalley)
Let $G \subset GL(V)$ be an algebraic group. **Then there exists a tensor representation W such that G is the stabilizer of a point $l \in \mathbb{P}W$.**

Proof. Step 1: Consider the algebra $\operatorname{Sym}^*(V \otimes V^*)$ of polynomial functions on $GL(V)$. We equip this algebra by $GL(V)$-action associated with the left translations of $GL(V)$ on itself. By definition, G is given as the set of common zeros of polynomials $P_1, P_2, ..., P_n \in \operatorname{Sym}^*(V \otimes V^*)$. Let d be the largest degree of P_i, and $W_1 := \oplus_{i=0}^d \operatorname{Sym}^i(V \otimes V^*)$. Denote by $W_2 \subset W_1$ the space of all polynomials of degree $\leqslant d$, vanishing on G. **Then G is a maximal subgroup of $GL(V)$ fixing W_2.** Indeed, an element $x \in GL(V)$, acting on $GL(V)$ by left translations, preserves W_2 if and only if it preserves the common zeroes of W_2, which is equivalent to $xG = G$, and this is equivalent to $x \in G$.

Step 2: Let $r := \dim W_2$, and $W := \Lambda^r W_1$. Consider the line $L := \Lambda^r W_2 \subset \Lambda^r W_1 = W$. Then $W_2 = \{v \in W_1 \ \ | \ \ v \wedge L = 0\}$. **Therefore, $x \in GL(V)$ preserves L if and only if x preserves W_2.**

Step 3: Consider the projectivization $\mathbb{P}W$. An element $x \in GL(V)$ fixes the point $\mathbb{P}L \in \mathbb{P}W \Leftrightarrow x$ preserves W_2 (Step 2) $\Leftrightarrow x$ belongs to G (Step 1). ■

Chevalley theorem for reductive groups

DEFINITION: An algebraic group G is **reductive** if its complexification $G_{\mathbb{C}}$ has a compact real form, that is, there exists a compact algebraic group K over $\mathbb{R}$ such that $G_{\mathbb{C}} = K \otimes_{\mathbb{R}} \mathbb{C}$.

REMARK: This is equivalent to semisimplicity of its category of representations.

THEOREM: Let $G \subset GL(V)$ be a reductive algebraic group. **Then there exists a tensor representation $\mathfrak{W}$ such that G is the stabilizer of a vector $w \in \mathfrak{W}$.**

Proof: By Chevalley's theorem, there exists a tensor representation W of $GL(V)$ such that G is the stabilizer of $L \in \mathbb{P}(W')$. Let $v \in L$ and $v^* \in L^*$ be non-zero vectors. Since G is reductive, $W = L \oplus W'$. Therefore, $L \otimes L^*$ is a direct summand of $W \otimes W^*$. **Clearly, $x \in GL(V)$ fixes $v \otimes v^*$ if and only if x fixes L.** We obtain Deligne's version of Chevalley theorem setting $w = v \otimes v^*$ and $\mathfrak{W} = W \otimes W^*$. ■

Hodge structures (reminder)

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb C} = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb Q}$ such that $V_{\mathbb R} = V_{\mathbb Q} \otimes \mathbb R$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb Z} \subset V_{\mathbb Q}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

REMARK: Rational structure on a real vector space V is a $\mathbb Q$-subspace $V_{\mathbb Q} \subset V$ such that $V = V_{\mathbb Q} \otimes_{\mathbb Q} \mathbb R$. **Integer structure** on a real vector space V is a $\mathbb Z$-sublattice $V_{\mathbb Z} \subset V$ such that $V = V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$.

REMARK: A real Hodge structure $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ on $V_{\mathbb R}$ is rational (integer) **if $V_{\mathbb R}$ is equipped with a rational (integer) structure.**

Polarization (reminder)

DEFINITION: Polarization on a rational Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$-\sqrt{-1}^{\,p-q} h(x, \overline{x}) > 0 \qquad (*)$$

("Hodge-Riemann relations") for each non-zero $x \in V^{p,q}$.

REMARK: $U(1)$-invariance of a pairing h means that the pairing of $V^{p,q}$ with V^{p_1,q_1} vanishes unless $p - q = q_1 - p_1$, or, equivalently, $p = q_1, q = p_1$.

DEFINITION: The objects of the **category of real (rational, integer) Hodge structures** are Hodge structures, morphisms are $\mathbb{R}$-linear ($\mathbb{Q}$-linear, $\mathbb{Z}$-linear) maps of vector spaces which preserve the Hodge decomposition.

Category of polarized Hodge structures

DEFINITION: The objects of the **category of rational or integer polarized Hodge structures are rational or integer Hodge structures admitting a polarization**, and morphisms are morphisms of Hodge structures. **The polarization is not fixed, the morphisms are not necessarily compatible with the polarization.**

DEFINITION: A **simple object** of an abelian category is an object which has no proper subobjects. An abelian category is **semisimple** if any object is a direct sum of simple objects.

CLAIM: Category of polarized Hodge structures in semisimple.

Proof: Orthogonal complement of a Hodge substructure $V' \subset V$ with respect to h is again a Hodge substructure, and this complement does not intersect V'; both assertions follow from the Hodge-Riemann relations. ∎

Mumford-Tate group

DEFINITION: Let V be a Hodge structure over $\mathbb{Q}$, and ρ the corresponding $U(1)$-action. **Mumford-Tate group** (Mumford, 1966; Mumford called it "the Hodge group") is the smallest algebraic group over $\mathbb{Q}$ containing ρ.

THEOREM: Let V be a rational, polarized Hodge structure, and $\operatorname{MT}(V)$ its Mumford-Tate group. Consider the tensor algebra of V, $W = T^{\otimes}(V)$ with the Hodge structure (also polarized) induced from V. Let W_h be the space of all ρ-invariant rational vectors in W (such vectors are called **"Hodge vectors"**). **Then $\operatorname{MT}(V)$ coincides with the stabilizer**

$$\operatorname{St}_{GL(V)}(W_h) := \{g \in GL(V) \quad | \quad \forall w \in W_h, \text{ one has } g(w) = w\}.$$

Proof: Follows from the theorem on tensor invariants for reductive groups. ■

Corollary 1: Let $V_{\mathbb{Q}}$ be a rational Hodge structure, and $W \subset V_{\mathbb{Q}}$ a subspace. **Then the following are equivalent.**
(i) W is a Hodge substructure.
(ii) $W \subset V_{\mathbb{Q}}$ is a Mumford-Tate invariant subspace. ■

Totally real fields and CM-fields

DEFINITION: A number field is a finite extension of $\mathbb{Q}$. A number field E is **totally real** if all its embedding to $\mathbb{C}$ are real (that is, their image belongs to $\mathbb{R}$), and **CM-type**, or **complex multiplication type** if E is quadratic extension of a totally real field E_0, but none of its embeddings to $\mathbb{C}$ are real.

EXAMPLE: A cyclotomic field $\mathbb{Q}[\zeta]$, with ζ being a primitive root of unity, is a CM-field. Indeed, $\mathbb{Q}[\zeta+\zeta^{-1}]$ **is totally real field.**

REMARK: Any Galois extension E of $\mathbb{Q}$ which admits a real embedding **is totally real.** Indeed, **the Galois group of E acts transitively on all embeddings of E to $\mathbb{C}$** and commutes with the complex conjugation, hence maps real embeddings to real embeddings.

THEOREM: (A. A. Albert)
Let M be an abelian variety with irreducible Hodge structure, and K the center of $\operatorname{End}_H(H^1(M,\mathbb{Q}))$, where End_H denotes the algebra of endomorphisms of rational Hodge structures. **Then K is a totally real field or a CM-field.**

Proof: Later today

The Rosatti involution

DEFINITION: Let $(V_{\mathbb{Q}}, V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, s)$ be a weight 1 polarized rational Hodge structure, and $\operatorname{End}_H(H^1(V))$ its automorphism algebra. **The Rosatti involution** $r \mapsto r^*$ takes $r \in \operatorname{End}_H(V)$ to a dual map with respect to the polarization.

REMARK: Since the polarization is rational, $r \mapsto r^*$ maps rational endomorphisms to rational. Since the polarization is $U(1)$-invariant, the Rosatti involution is compatible with the Hodge structure on $\operatorname{End}(V)$, and maps $U(1)$-invariant endomorphisms to $U(1)$-invariant ones. Therefore, **the Rosatti involution takes $\operatorname{End}_H(V)$ to itself.**

CLAIM: Consider a bilinear symmetric form form on $\operatorname{End}_H(V))$ taking x to $\operatorname{Tr}_V(xx^*)$, considered as an element of $\mathbb{Q}$. **Then $\operatorname{Tr}_V(xx^*) > 0$ for all $x \neq 0$.**

Proof: By construction, $r \mapsto r^*$ is the Hermitian conjugation of a complex-linear matrix; however, $\operatorname{Tr}(A\overline{A}^T) > 0$ for any $A \in GL(n, \mathbb{C})$. ■

Albert's theorem

THEOREM: (A. A. Albert)
Let M be an abelian variety with irreducible Hodge structure, and K the center of $\operatorname{End}_H(H^1(M,\mathbb{Q}))$. **Then K is a totally real field or a CM-field.**

Proof. Step 1: If the Rosatti involution acts on K by identity, we have $\operatorname{Tr}_V(x^2) > 0$ for any element of K. Tensoring K with $\mathbb{R}$, we obtain a positive definite $\mathbb{R}$-linear form Tr_V on $K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r \oplus \mathbb{C}^s$, which also satisfies $\operatorname{Tr}_V(x,y) = \varepsilon(xy)$ for some $\mathbb{R}$-linear functional $\varepsilon : (K \otimes \mathbb{R}) \longrightarrow \mathbb{R}$. On $\mathbb{C}$, such 2-form does not exist, because $t \mapsto t^2$ is surjective. We have shown that **$K \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^r$, and K is totally real** whenever the Rosatti involution is identity.

Step 2: Assume that the Rosatti involution is non-trivial, and let K_0 be its fixed set. Since the square of any element of K belongs to K_0, the extension $[K : K_0]$ is quadratic, $K = K_0[\sqrt{a}]$, and Rosatti acts as $(\sqrt{a})^* = -\sqrt{a}$. **Since $\operatorname{Tr}_V(\sqrt{a}(\sqrt{a})^*) = \operatorname{Tr}_V(-a) > 0$, this implies that K has no real embeddings.** ■

Transcendental Hodge lattice

DEFINITION: Let $V_{\mathbb{Q}}$ be a rational, polarized Hodge structure of weight d, $V_C = \bigoplus_{\substack{p+q=d \\ p,q\geqslant 0}} V^{p,q}$. Then a minimal Hodge substructure $V^{tr} \subset V_{\mathbb{C}}$ containing $V^{d,0}$ is called **the transcendental Hodge lattice**.

THEOREM: Transcendental Hodge lattice is a birational invariant.

Proof: Let $\varphi : X \longrightarrow Y$ be a birational morphism of projective varieties. Then $\varphi^* : H^d(Y) \longrightarrow H^d(X)$ induces isomorphism on $H^{d,0}$. Therefore, it is injective on $H^d_{tr}(Y)$. Indeed, its kernel is a Hodge substructure of $H^d_{tr}(Y)$ not intersecting $H^{d,0}$, which is impossible. Applying the same argument to the dual map, we obtain that φ^* is also surjective on $H^d_{tr}(Y)$. ■

Hodge structures of K3 type

DEFINITION: A polarized, rational Hodge structure $V_{\mathbb{C}} = \bigoplus_{\substack{p+q=2 \\ p,q\geqslant 0}} V^{p,q}$ of weight 2 with $\dim V^{2,0} = 1$ is called **a Hodge structure of K3 type**.

DEFINITION: A Hodge structure is called **simple** if it has no proper Hodge substructures.

REMARK: Let M be a projective K3 surface, and $V_{\mathbb{Q}}$ its transcendental Hodge lattice. **Then it is simple and of K3 type.**

PROPOSITION: (Zarhin) Let $V_{\mathbb{Q}}$ be a simple Hodge structure of K3 sype, and $E = \operatorname{End}(V_{\mathbb{Q}})$ an algebra of its endomorphisms in the category of Hodge structures. **Then E is a number field.**

Proof: By Schur's lemma, E has no zero divisors, hence it is a division algebra. Since $E \subset \operatorname{End}(V_{\mathbb{Q}})$, it is countable. **To prove that it is a number field, it remains to show that E is commutative.** However, E acts on a 1-dimensional space $V^{2,0}$. This defines a homomorphism from E to $\mathbb{C}$, which is injective, because E is a division algebra. ■

Hodge structures of K3 type (2)

THEOREM: (Zarhin) Let $V_{\mathbb{Q}}$ be a Hodge structure of K3 type, and $E := \operatorname{End}(V_{\mathbb{Q}})$ its endomorphism field. **Then E is either totally real (that is, all its embeddings to $\mathbb{C}$ are real) or is a CM-field.**

Proof. Step 1: Let $a \in E$ be an endomorphism, and a^* its conjugate with respect to the polarization h. Since the polarization is rational and $U(1)$-invariant, the map a^* also preserves the Hodge decomposition. Then $a^* \in E$. Denote the generator of $V^{2,0}$ by Ω. Then $h(a(\Omega), a(\overline{\Omega})) = h(a^*a(\Omega), \overline{\Omega}) > 0$, hence $\frac{a^*a(\Omega)}{\Omega}$ is a positive real number. Then, the embedding $E \hookrightarrow \mathbb{C}$ induced by $b \longrightarrow \frac{b(\Omega)}{\Omega}$ maps a^*a to a positive real number.

Step 2: The group $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$ maps the Hodge structure to another Hodge structure, and $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$ acts transitively on embeddings from a Galois closure of E to $\mathbb{C}$. This group acts on E and commutes with the map $a \to a^*$. Therefore, **all embeddings $E \hookrightarrow \mathbb{C}$ map a^*a to a positive real number.**

Step 3: The map $a \longrightarrow a^*$ is either a non-trivial involution or identity. In the second case, all embeddings $E \hookrightarrow \mathbb{C}$ map a^2 to a positive real number, and E is totally real. In the first case, the fixed set $E^\tau =: E_0$ is a degree 2 subfield of E, with τ the generator of the Galois group $\mathcal{G}al(E/E_0)$. ■

Zarhin's results about Mumford-Tate group

THEOREM: (Zarhin) Let $V_{\mathbb{Q}}$ be an irreducible Hodge structure of K3 type, and $E := \operatorname{End}(V_{\mathbb{Q}})$ the corresponding number field. Denote by $SO_E(V)$ the group of E-linear isometries of V for $[E : \mathbb{Q}]$ totally real, and by $U_E(V)$ the group of E-linear isometries of V for E an imaginary quadratic extension of a totally real field. **Then the Mumford-Tate group MT is $SO_E(V)$ in the first case, and $U_E(V)$ in the second.**

Proof: *Zarhin, Yu.G.,* Hodge groups of K3 surfaces, *Journal für die reine und angewandte Mathematik Volume 341, page 193-220, 1983.* ■