Variations of Hodge structures

lecture 7: Noether-Lefschetz theorem

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Hodge structures (reminder)

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb C} = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb Q}$ such that $V_{\mathbb R} = V_{\mathbb Q} \otimes \mathbb R$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb Z} \subset V_{\mathbb Q}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

REMARK: Rational structure on a real vector space V is a $\mathbb Q$-subspace $V_{\mathbb Q} \subset V$ such that $V = V_{\mathbb Q} \otimes_{\mathbb Q} \mathbb R$. **Integer structure** on a real vector space V is a $\mathbb Z$-sublattice $V_{\mathbb Z} \subset V$ such that $V = V_{\mathbb Z} \otimes_{\mathbb Z} \mathbb R$.

REMARK: A real Hodge structure $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ on $V_{\mathbb R}$ is rational (integer) **if $V_{\mathbb R}$ is equipped with a rational (integer) structure.**

Variations of Hodge structures (reminder)

DEFINITION: Let M be a complex manifold. A **variation of Hodge structures (VHS)** on M is a complex vector bundle (B, ∇) with a flat connection equipped with a parallel anti-complex involution and a Hodge structure, $B = \bigoplus_{p+q=w} B^{p,q}$ which satisfy **"Griffiths transversality condition":** $\nabla^{1,0}(B^{p,q}) \subset B^{p,q} \oplus B^{p+1,q-1}$.

DEFINITION: A **polarized VHS** (integer, rational VHS) is a VHS (B, ∇), $B = \bigoplus_{p+q=w} B^{p,q}$ such that ∇ preserves the polarization and the integer or rational lattice.

EXAMPLE: Let $\pi: M \longrightarrow X$ be a proper holomorphic surjective submersion. Consider the bundle $V := R^k\pi_*(\mathbb{C}_M)$ with the fiber in x the k-th cohomology of $\pi^{-1}(x)$, the Hodge decomposition coming from the complex structure on $\pi^{-1}(x)$, and the Gauss-Manin connection. **This defines a variation of Hodge structures**.

REMARK: Consider a term

$$F_{d+1} := B^{p-r,q} \oplus B^{p-r-1,q+1} \oplus ... \oplus B^{p-r-d,q+d}$$

of the Hodge filtration. **Then $F_{d+1} \subset B$ is a holomorphic sub-bundle.**

Proof: As we have already seen, $\nabla_{\theta^{0,1}} F_{d+1} \subset F_{d+1}$. ■

Fixed part theorem (reminder)

THEOREM: (Deligne-Griffiths-Schmid's fixed part theorem)
Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base, and b a parallel section of B. **Then all (p,q)-components of b are also parallel.**

Proof: Lecture 4. ■

REMARK: This statement **also holds when M is quasiprojective (we prove it later in this course, if time permits).**

COROLLARY: Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base. Assume that the monodromy of ∇ is trivial. **Then the Hodge decomposition is preserved by ∇**, that is, **the corresponding variation of Hodge structures is constant.**

Proof: Consider a basis $e_1, ..., e_n$ of $B|_x$ such that each e_i belongs to some $B^{p,q}$. Since the monodromy of ∇ is trivial, we can extend each e_i to a parallel section $\tilde{e}_i$ of B. The Hodge components of each of $\tilde{e}_i$ are parallel, but it has only one Hodge component at x. **Therefore, $\tilde{e}_i$ has only one Hodge component at each point of M**, and the corresponding Hodge decomposition is also constant. ■

Deligne's semisimplicity theorem (reminder)

DEFINITION: We further weaken the notion of real Hodge structures, defining **a complex Hodge structure on a vector space A**, which is a decomposition $A = \bigoplus A^{p,q}$, without the assumption $\overline{A^{p,q}} = A^{q,p}$. **A complex VHS** is a decomposition $V = \bigoplus V^{p,q}$ of a flat vector bundle (V, ∇) such that ∇ acts on $V = \bigoplus V^{p,q}$ satisfying the Griffiths transversality.

THEOREM: (Deligne's semisimplicity theorem)
Let $(V, \nabla, V = \bigoplus V^{p,q})$ be an integer, polarized VHS over a compact (or quasiprojective) base. **Then the flat bundle (V, ∇) can be decomposed as $V = \oplus_i L_i \otimes_{\mathbb{C}} W_i$, where L_i are flat bundles with irreducible monodromy, and W_i complex vector spaces.** Moreover, **each L_i is equipped with a structure of a complex VHS, and each W_i with a complex Hodge structure,** in such a way that this decomposition is compatible with the Hodge structures.

Proof: Lecture 5.

REMARK: This decomposition **is not necessarily compatible with the integer (or even rational) structure** on V.

Noether-Lefschetz theorem

DEFINITION: Let M be a compact Kähler manifold. Denote the lattice $H^{1,1}(M) \cap H^2(M,\mathbb{Z})$ by $NS(M)$. This lattice is called **Picard lattice**, or **Neron-Severi lattice**. The number $\operatorname{rk} NS(M)$ is called **the Picard rank** of M.

THEOREM: Let X be a general hypersurface of degree $d \geqslant 4$ in $\mathbb{C}P^n$, with $n = 3$. **Then its Picard rank is 1.**
Proof: Later today.

REMARK: This statement is false for $d = 3$, $n = 3$. Indeed, a smooth cubic surface in $\mathbb{C}P^3$ is $\mathbb{C}P^2$ blown up in 6 points (Clebsch), which gives $\operatorname{rk} NS(X) = 7$.

REMARK: This statement is false for $d = 2$, $n = 3$. Indeed, a smooth quadric in $\mathbb{C}P^3$ is $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Noether-Lefschetz theorem (2)

REMARK: Consider the Veronese embedding V of $\mathbb{C}P^n$ to $\mathbb{P}(H^0(\mathbb{C}P^n, \mathcal{O}(d))^*)$ given by the space of all degree d polynomials. Then X is a hyperplane section of $\operatorname{im} V$. By Lefschetz' hyperplane section theorem, the embedding map $X \longrightarrow \operatorname{im} V$ induces an isomorphism $H^k(\operatorname{im} V) \cong H^k(X)$ for all $k < d$, and this isomorphism is compatible with the Hodge structure. **Therefore, $NS(X) = NS(\mathbb{C}P^n)$ when $n > 3$. The only non-trivial case of Noether-Lefschetz theorem is when $n = 3$, and X is a complex surface**.

THEOREM: (T. Shioda)
For any prime $p \geqslant 5$, the surface in $\mathbb{C}P^3$ given by an equation $w^p + xw^{p-1} + yz^{p-1} + zx^{p-1} = 0$ **has Picard rank 1.**

Proof: Tetsuji Shioda, *ON THE PICARD NUMBER OF A COMPLEX PROJECTIVE VARIETY*, Ann. scient. EC. Norm. Sup. 4e serie, v. 14, 1981, pp. 303-321. ■

REMARK: For smooth quartics in $\mathbb{C}P^3$ (which are all K3 surfaces), **it is much harder to find an explicit equation for a quarting with has Picard rank 1**. This question, due to Mumford, was open for almost 30 years, until early 2000-ies.

Period space (reminder)

DEFINITION: Let V be a real vector space, and $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure. Assume that $p,q \geqslant r$ **The Hodge filtration** is the following filtration on the vector space $V_{\mathbb C}$:

$$0 \subset V^{r,w-r} \subset V^{r,w-r} \oplus V^{r+1,w-r-1} \subset V^{r,w-r} \oplus V^{r+1,w-r-1} \oplus V^{r+2,w-r-2} \oplus ...$$

Denote by F_n the n-th term of this filtration, $F_n := \bigoplus_{i=0}^{n-1} V^{r+i,w-r-i}$. Clearly, $V^{p,w-p} = F_{p-r+1} \cap \overline{F}_{w-p-r+1}$ **(prove this)**, hence **the Hodge filtration determines the Hodge structure uniquely.**

REMARK: Two subspaces $W_1, W_2 \subset V$ intersect transversally when $W_1 + W_2 = V$. **Therefore, F_{p-r+1} and $\overline{F}_{w-p-r+1}$ intersect transversally.** **DEFINITION:** Let $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$ a Hodge structure on V. Fix dimensions of all $V^{p,q}$; this determines the dimensions of F_i. **The period space** is the space of all flags $0 \subset F_1 \subset ... \subset F_{w-2r} = V$ such that F_{p-r+1} and $\overline{F}_{w-p-r+1}$ intersects transversally for all p.

CLAIM: The points in the period space **are in bijective correspondence with the set of all Hodge structures on V** having the same numbers $\dim V^{p,w-p}$. ■

REMARK: The period space **is an open subset in the corresponding partial flag space**, which is considered as a complex projective manifold.

Period map (reminder)

DEFINITION: Let M be a simply connected complex manifold, and $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ a variation of Hodge structures. Since ∇ is flat, the corresponding local system is trivial, and parallel transport identifies all fibers of B and trivializes B. The **period map** is a map taking $m \in M$ to the corresponding point $0 \subset F_1|_m \subset ... \subset F_{w-2r}|_m = B|_m$ in the period space $\mathbb{P}\mathrm{er}$.

CLAIM: The period map $\mathrm{Per} : M \longrightarrow \mathbb{P}\mathrm{er}$ is holomorphic.
Proof: The Hodge filtration is holomorphic, hence **the map which associates to a point $m \in M$ a subspace $F_i|_m \subset B|_m$ is also holomorphic.** Indeed, locally F_i has a holomorphic basis $f_1, ..., f_k$, and the corresponding Plücker map can be expressed as $m \mapsto f_1 \wedge ... \wedge f_k$, where $f_1 \wedge ... \wedge f_k$ is considered as an element of $\mathbb{P}\Lambda^k B = M \times \mathbb{P}\Lambda^k B|_m$. ■

Noether-Lefschetz loci (reminder)

DEFINITION: Let M be a simply connected complex manifold, and $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ a variation of Hodge structures. Assume that $\pi_1(M) = 0$, Since ∇ is flat, the corresponding local system is trivial, and parallel transport identifies all fibers of B and trivializes B. Denote by $B_{\mathbb{R}} \subset B$ the set of fixed points of the anticomplex involution on B. Fix a subspace $V \subset B_{\mathbb{R}}|_x$. **Noether-Lefschetz locus** associated with V is the set of all $x \in X$ such that $V \subset B^{u,u}|_x$, where $u = w/2$.

THEOREM: The Noether-Lefschetz locus **is a complex subvariety of M.**

Proof. Step 1: Let $F_{\text{middle}} := \bigoplus^{p,q}_{p \geqslant q}$. Clearly, F_{middle} is a component of the Hodge filtration. Since V is real, and $F_{\text{middle}} \cap \overline{F}_{\text{middle}} = B^{u,u}$, **the Noether-Lefschetz locus is the set of all $x \in M$ such that $V \subset F_{\text{middle}}|_x$.**

Step 2: Let $f_1, ..., f_k$ be a holomorphic basis in F_{middle}. Denote by $f \in \Lambda^k B$ the vector $f_1 \wedge ... \wedge f_k$. For each $v \in V$, the set N_v of all $x \in M$ such that $v \in F_{\text{middle}}|_x$ **is the zero set of a holomorphic section $v \wedge f \in \Lambda^{k+1} B$,** hence it is a complex subvariety in M. Now, the Noether-Lefschetz locus is $\bigcup_{v \in V} N_v$.
■

Irreducibility of the monodromy for the universal family of degree d surfaces

Let $S = H^0(\mathbb{C}P^3, \mathcal{O}(d))$ be the space of all homogeneous polynomials of 4 variables of degree d, and $S_0 \subset S$ an open subset which corresponds to polynomials which give smooth surfaces of degree d in $\mathbb{C}P^3$. Consider the incidence variety $Z \subset \mathbb{C}P^3 \times \mathbb{P}S$ formed by all pairs

$$\{(z, f) \in \mathbb{C}P^3 \times \mathbb{P}S \quad f(z) = 0\}$$

Clearly, Z is fibered over $\mathbb{P}S$ with the fibers at $f \in \mathbb{P}S$ isomorphic to the corresponding surface of degree d. Let Z_0 be the preimage of S_0 in Z. By construction, **the fibration $Z_0 \longrightarrow S_0$ is a smooth, proper submersion with projective fibers.** Recall that "primitive part" of $H^2(S)$, for a projective complex surface, is the orthogonal complement to the Kähler class, taken with respect to the intersection form; this is the space equipped with the polarized Hodge structure.

THEOREM: The monodromy of the Gauss-Manin connection associated with the primitive second cohomology of the fibers of $Z_0 \longrightarrow S_0$ **is irreducible.**

Proof: Not today. The proof uses Lefschetz pencils, vanishing cycles and Picard-Lefschetz theory.

Noether-Lefschetz theorem and irreducibility of monodromy

THEOREM: Let X be a general hypersurface of degree $d \geqslant 4$ in $\mathbb{C}P^n$, with $n = 3$. **Then its Picard rank is 1.**

Proof. Step 1: Let $Z \subset \mathbb{C}P^3 \times \mathbb{P}S$ be the incidence variety constructed above, and $f: Z_0 \longrightarrow S_0$ its smooth locus. Consider all elements of Picard lattice which remains of type (1,1) in the local system V of second cohomology of fibers. This is a local subsystem; since V is irreducible, this subsystem is empty (and in this case the Noether-Lefschetz loci have positive codimension), or it is everyting.

Step 2: In the second case, $H^{2,0}(F) = 0$ for a smooth fiber of f. However, the adjunction formula gives $K_F = K\mathbb{C}P^2 \otimes N_F = \mathcal{O}(-4) \otimes O(d) = \mathcal{O}(d-4)$, and it has sections when $d \geqslant 4$. ■