Variations of Hodge structures

lecture 8: Lefschetz pencils

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Lefschetz pencils

REMARK: Let $f \in \mathcal{O}_M$ be a holomorphic function on a complex manifold, and $x \in M$ its critical point. **The Hessian** is the matrix of second derivatives of f.

EXERCISE: Prove that **the Hessian form** $\operatorname{Hess}(f) \in \operatorname{Sym}^2(T_x^* M)$ **is independent from the choice of coordinates.**

DEFINITION: Let $X \subset M$ be a hypersurface locally given by an equation $f = 0$, for a holomorphic function $f \in \mathcal{O}_M$, and $x \in X$ its singular point. Then $df\Big|_{T_xM} = 0$. We say that x is **an ordinary double point of X** if $\operatorname{Hess}(f)$ is a non-degenerate 2-form on T_xM.

DEFINITION: Let L be a holomorphic line bundle on a compact complex manifold M, and $W \subset H^0(M, L)$ a 2-dimensional subspace. The zero sets of $x \in W\backslash 0$ is a collection D_t of divisors parametrized by $\mathbb{P}H^0(M, L) = \mathbb{C}P^1$. It is called **a pencil of divisors**. Its **base locus** is the intersection of all $D_t, t \in \mathbb{C}P^1$.

DEFINITION: **A Lefschetz pencil** is a pencil of divisors such that its base locus B is smooth and all D_t are smooth along B, and all singularities of D_t are ordinary double points.

Projective dual variety

Let $\mathbb{P}^n = \mathbb{P}V$ be a complex projective space. We denote $\mathbb{P}V^*$ by $\check{\mathbb{P}}^n$, and **identify it with the set of all projective hyperplanes in $\mathbb{P}^n$.**

DEFINITION: Let $X \subset \mathbb{P}^n$ be a complex manifold. **The projectively dual variety** $X^\vee$ is the space of all hyperplanes $V \in \check{\mathbb{P}}^n$ tangent to some point in X.

DEFINITION: Consider the set $P_X \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ of all pairs $x \in X, W \in \check{\mathbb{P}}^n$ such that W is tangent in x to X. The variety P_X is called **projectivised conormal bundle**. Clearly, $X^\vee$ is the image of P_X under the projection map $\pi: \mathbb{P}^n \times \check{\mathbb{P}}^n \longrightarrow \check{\mathbb{P}}^n$.

REMARK: P_X is the set of all pairs $(x \in C, \lambda \in \check{\mathbb{P}}^n)$ such that the corresponding functional $\lambda^\circ \in V^*$ vanishes on $T_x C$. In other words, $\lambda^\circ \in (T\mathbb{P}^n/TX)^*$. **This is why P_X is identified with the projectivised conormal bundle to $X \subset \mathbb{P}^n$.**

DEFINITION: Let $S \subset \mathbb{P}^n$ be a smooth hypersurface. **The Gauss map** takes a smooth point $x \in S$ to the tangent space $T_x S \in \check{\mathbb{P}}^n$.

REMARK: If X is a hypersurface, **$X^\vee$ is its image under the Gauss map.**

Conical Lagrangial varieties

DEFINITION: A Lagrangian subvariety in a holomorphic symplectic manifold is a subvariety which is Lagrangian in all its smooth points.
DEFINITION: Let $V = \mathbb{C}^n$ be a vector space, considered as a holomorphic symplectic manifold and $L \subset V \times V^*$ a Lagrangian subvariety. It is called **conical** if it is invariant under dilation of V^*.

EXAMPLE: Let $X \subset V$ be a complex subvariety, and C_X its conormal bundle, that is, the closure of the set of all $(x, y) \in V \times V^*$ such that x is smooth and $y|_{T_x}X = 0$. **Clearly, C_X is a conical Lagrangian subvariety.**

CLAIM: Any irreducible conical Lagrangian subvariety $L \subset V \times V^*$ is obtained this way.

Proof: Consider the projection $\pi : L \longrightarrow V$. Since L is conical, its image is also the image of its projectivization, which is proper, hence $Y := \pi(L)$ is a complex subvariety of V. Since C_Y and L have the same dimension, to prove that $L = C_Y$ it would suffice to show that $L \subset C_Y$. Equivalently, **we need to prove that for any $(v, \nu) \in L$, we have $v \in Y$, and $\nu|_{T_vY} = 0$.** The first is clear from the definition. To prove the second, we notice that vector $(0, \nu) \in T_{(v,\nu)}(V \times V^*)$ is tangent to the dilatation, hence belongs to $T_{(v,\nu)}L$. When $v \in Y$ is smooth, the projection $d\pi : T_{(v,\nu)}L \longrightarrow T_vY$ is sutjective. Since $(0, \nu)$ is symplectic orthogonal to any $(x, 0) \in T_{(v,\nu)}L$, this gives $\nu|_{T_vY} = 0$ for any smooth point $v \in Y$. ■

Lagrangian duality

DEFINITION: Let $A \subset V$ be a $\mathbb{C}^*$-invariant algebraic subvariety, and $C_A \subset V \times V^*$ its conormal variety. Consider C_A as a conical Lagrangian subvariety of $V^* \times V$. Then C_A is a conormal subvariety of $A^\vee \subset V^*$. The variety $A^\vee$ is called **the Lagrangian dual** of A.

PROPOSITION: $(A^\vee)^\vee = A$**.**

Proof: Let $C_A^\vee$ be the subset of $V^* \times V$ obtained from C_A by permutation of two components. Since A is $\mathbb{C}^*$-invariant, $C_A^\vee$ is a conical Lagrangian subvariety, **hence $C_A^\vee$ is the conormal variety of $\pi(C_A^\vee) = A^\vee$, that is, $C_A^\vee = C_{A^\vee}$.** Then the second dual of $A^\vee$ is obtained by projecting $C_A^\vee$ to the second component; by definition, this gives A again. ■

REMARK: The idea to deduce projective duality from Lagrangian duality is due to *Evgueni Tevelev, Projectively Dual Varieties, arXiv:math/0112028.*

Projective duality theorem

DEFINITION: Open affine cone of a projective manifold $X \subset \mathbb{P}V$ is its preimage $X^\circ \subset V$ under the projection map $V\backslash 0 \longrightarrow \mathbb{P}V$. **Closed affine cone** is its closure, which is an affine subvariety in V.

REMARK: Let $X \subset \mathbb{P}V$ be a projective variety, and $X^\circ \subset V$ its affine cone. The projection $\pi(C_{X^\circ})$ to the second argument is a $\mathbb{C}^*$-invariant subset of V^*. By definition, **its closure coincides with $X^\vee$.**

THEOREM: Let $X \subset \mathbb{P}V$ be a projective variety. **The double dual of X is X again, $(X^\wedge)^\wedge = X$.**

Proof: The dual of X is projectivization of the Lagrangian dual to the cone X°, hence **the second projective dual is the projectivization of the Lagrangian second dual to X°.** However, the Lagrangian second dual to X° is X°, as shown above. ■

COROLLARY: The Gauss map $G: X \longrightarrow X^\vee$ **is a local diffeomorphism in all points $a \in X$ such that $G(a)$ is a smooth point on $X^\vee$.**

Proof: The composition of Gauss maps $X \xrightarrow{G} X^\vee \xrightarrow{G} X$ is identity in all points where it is well defined. ■

Lines transversal to $X^\vee$

Let $A = \mathbb{C}P^{n-2} \subset \mathbb{P}^n$ be a projective subspace, embedded in the standard way, and $Z \subset \check{\mathbb{P}}^n$ the set of all planes containing A. Clearly, Z is a pencil of divisors. **We identify Z with its image $L = \mathbb{C}P^1 \subset \check{\mathbb{P}}^n$.**

Fix a closed, irreducible, smooth $X \subset \mathbb{P}^n$. **The intersection of Z with X gives a pencil of divisors on X with the base set $Y = A \cap X$.**

DEFINITION: The pencil $L \subset \check{\mathbb{P}}^n$ **is transversal to $X^\vee$** if L meets $X^\vee$ only in its smooth part, and is transversal to $X^\vee$ in all intersection points.

CLAIM: A general line $L \subset \check{\mathbb{P}}^n$ is transversal to $X^\vee$.

Proof: The set of all lines in $\mathbb{P}^n$ is $2n-2$-dimensional: a line is determined by two points in $\mathbb{P}^n$, and the space of pairs of points on $\mathbb{C}P^1$ is 2-dimensional. The singular set $X^\vee_{\text{sing}}$ of $X^\vee$ is at most $n-2$-dimensional, hence the set of all lines meeting $X^\vee_{\text{sing}}$ is at most $2n-3$-dimensional. The set of all lines tangent to a given smooth point $x \in X^\vee$ is $n-2$-dimensional, hence the set of all lines tangent to $X^\vee$ at some smooth points has dimension $\dim X^\vee + n - 2 = 2n - 3$.
■

Existence of Lefschetz pencils

DEFINITION: A Lefschetz pencil is a pencil of divisors such that its base locus Y is smooth and all D_t are smooth along Y, and all singularities of D_t are ordinary double points.

THEOREM: Let $X \subset \mathbb{P}^n = \mathbb{P}^n V$ be a submanifold, and $L \subset \check{\mathbb{P}}^n = \mathbb{P}^n V^*$ a line which is transversal to $X^\vee$. **Then L defines a Lefschetz pencil on X.**

Proof. Step 1: Let A be the base point of L considered as a pencil of hyperplanes in $\mathbb{P}^n$. **Then A is transversal to X.** Indeed, suppose that A is not transversal to X in $a \in X$. Then there is a hyperplane $H \in \check{\mathbb{P}}^n$ containing A and tangent to X in a. The point of $\check{\mathbb{P}}^n$ corresponding to this hyperplane belongs to $L \cap X^\vee$. Since L is tranversal to $X^\vee$, the space $X^\vee$ is smooth in the point $H = G(a) \in L \cap X^\vee$. By projective duality, this implies that $G(H) = a$, where $G: X^\vee \longrightarrow \mathbb{P}^n$ denotes the Gauss map. Therefore, H is the plane in $\check{\mathbb{P}}^n$ associated with a. Since the divisors associated to all points $l \in L$ contain a, **by duality $L \subset H$, and L cannot be transveral to $X^\vee$ in a.**

Step 2: It remains only to show that the singularities of the divisors $D_t := H_t \cap X$, $H_t \in L$ **are ordinary double points.**

Existence of Lefschetz pencils (2)

Step 2: It remains only to show that the singularities of the divisors $D_t := H_t \cap X$, $H_t \in L$ **are ordinary double points.**

Step 3: Let $a \in H_t$ be a critical point, G the Gauss map, and $G(a) \in X^\vee$ the corresponding point in $X^\vee$. Since H_t and X are not transversal in a, the embedding $T_a(H_t \cap X) \longrightarrow T_a X$ is not strict, and $T_a H_t = T_a X$. This implies that L passes through $G(a)$, where G is the Gauss map. Since L is transversal to $X^\vee$ in H_t, $X^\vee$ is smooth in a, hence $G: X^\vee \longrightarrow X$ **composed with** $G: X \longrightarrow X^\vee$ **is identity, by projective duality theorem, and** G **is invertible.**

Step 4: Consider n functions $f_i: \mathbb{C}^{n-1} \longrightarrow \mathbb{C}$, depending on variables $t_1, ..., t_{n-1}$ Let $1 : z_1 : ... : z_n$ be an affine chart in $\mathbb{P}^n$, and $1 : f_1 : ... : f_n$ a hypersurface $Z = F(t_1, ..., t_{n-1})$, parametrized by $t_1, ..., t_{n-1}$. The cone of Z is parametrized by $(t, tf_1, ..., tf_n)$, and its tangent plane is generated by $(1, f_1, ..., f_n)$ and $(0, df_1(u), ..., df_n(u))$. where $u \in \langle d/dt_1, ..., d/dt_n \rangle$. Then $G(F(t_1, ..., t_{n-1}))$ is a subspace in V^* generated by $df_1(u), ..., df_n(u)$ and $(1, f_1, ..., f_n)$. After passing to the affine coordinate system, **we obtain that** $G(F(t_1, ..., t_{n-1}))$ **is a space tangent to** $1 : f_1 : ... : f_n$ **and generated by** $df_1(u), ..., df_n(u)$**.**

Step 5: We choose an affine chart $\mathbb{A}^n$ on $\mathbb{P}^n$ around a in such a way that H_t is a coordinate plane, and X is given as a graph of a function $(f(z_2, ..., z_n), z_2, ..., z_n)$. Then the Gauss map $G: X \longrightarrow \check{\mathbb{P}}^n$ takes the point $v := (f(z_2, ..., z_n), z_2, ..., z_n)$ to the hyperplane $\langle df(u), d/dz_2, ..., d/dz_n \rangle \subset T_v\mathbb{A}^n$ We consider $G(X)$ as a subvariety in $\check{\mathbb{P}}^n$ parametrized by $z_2, ..., z_n$. As follows from Step 2, the Gauss map takes the hyperplane $\check{\mathbb{P}}^n$ $\langle df(u), d/dz_2, ..., d/dz_n \rangle \subset T_v\mathbb{A}^n$ (considered as a point in $\check{\mathbb{P}}^n$) to Hess(f) evaluated in an $n-1$-tuple $(d/dz_2, ..., d/dz_n)$; However, the Gauss map is invertible, hence the rank of $dG(v)$ is maximal possible; this implies that $\det \operatorname{Hess} f \neq 0$. **We have shown that all singularities of D_t are ordinary double points.** ■