Variations of Hodge structures

lecture 9: Picard-Lefschetz theory

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Milnor fibration

Claim 1: Let $q: \mathbb{C}^n \longrightarrow \mathbb{C}$ be a non-degenerate quadratic form, and B_ε a closed ball with center in 0. Denote by V_t the intersection $q^{-1}(t) \cap B$. **Then $q^{-1}(t)$ is diffeomorphic to the total space of the tangent bundle TS^{n-1}.** Moreover, for $0 < t < \varepsilon$, **the manifold V_t transversally intersects the boundary of the ball B_ε, and the intersection $B_\varepsilon \cap q^{-1}(t)$ is diffeomorphic to a bundle of unit balls in TS^{n-1}.**

Proof. Step 1: We prove a diffeomorphism between $q^{-1}(t)$ and TS^{n-1} for $t = 1$; for all other t this diffeomorphism is obtained by rescaling. Let $z_i = x_i + \sqrt{-1}\, y_i$ be the complex and real coordinates on $\mathbb{C}^n$. Then

$$V_1 = q^{-1}(1) = \left\{ (x_1, y_1, ..., x_n, y_n) \ \middle|\ \sum_{i=1}^n x_i y_i = 0, \quad \sum x_i^2 - \sum y_i^2 = 1 \right\}.$$

We rewrite the last equation as $\sum x_i^2 = 1 + \sum y_i^2$. Then

$$(x'_1, ..., x'_n) := \left(\sqrt{(1 + \sum y_i^2)}\right)^{-1} (x_1, ..., x_n)$$

belongs to the unit sphere. In other words, **V_1 is the set of pairs $X, Y \in \mathbb{R}^n$ such that $X = X'\sqrt{1 + |Y|^2}$, where $X' = (x'_1, ..., x'_n)$ is a point in the unit sphere, and $Y = (y_1, ..., y_n) \in (x'_1, ..., x'_n)^\perp$.** For any point $p \in \mathbb{R}^n$ on a sphere S^{n-1}, one has $p^\perp = T_p S^{n-1}$, which gives $V_1 \cong TS^{n-1}$.

Milnor fibration (2)

Step 2: Instead of transversality of V_t and the unit sphere for $|t| < 1$, we prove the transversality of V_1 and the sphere of radius r. As follows from Step 1, the set V_1 is the set of pairs $(X, Y) \in (\mathbb{R}^n)^2$ with $Y = (y_1, ..., y_n) \in (x'_1, ... x'_n)^\perp$ and $X = (x'_1, ... x'_n)\sqrt{1 + |Y|^2}$, where $X' = (x'_1, ... x'_n) \in S^{n-1}$. This set is transversal to a sphere S_r of radius r in $z \in S_r$ when the radius-vector $\rho := \sum_i x_i \frac{d}{dx_i} + y_i \frac{d}{dy_i} \in T_z\mathbb{C}^n$ is not Euclidean orthogonal to T_zV_1. However, $|X|^2 + |Y|^2 = 1 + 2|Y|^2$. **Multiplying Y by $1+\delta$ takes $|X|^2 + |Y|^2 = 1 + 2|Y|^2$ to $1 + 2(1 + \delta)^2|Y|^2$, hence T_zV_1 is not orthogonal to the radius-vector in any $z \in V_1$ such that $|Y| > 0$.** If $|Y| = 0$, the radius r of the sphere is $|X| = 1$.

Step 3: Let $t \in \mathbb{R}^{>0}$. Rescaling, we obtain that the intersection of B_ε and V_t for $0 < t < \varepsilon$ is diffeomorphic to $V_1 \cap B_r$, for $r > 1$. Step 1 implies that V_1 is the set of all $(X, Y) \in (\mathbb{R}^n)^2$ with $Y \in X^\perp$, such that $|X| = \sqrt{1 + |Y|^2}$. Therefore, $V_1 \cap B_r$ is the set of all $(X, Y) \in (\mathbb{R}^n)^2$ with $Y \in X^\perp$, such that $|X| = \sqrt{1 + |Y|^2}$ and $|(X, Y)|^2 = 1 + 2|Y|^2 \leqslant r$. This is rewritten as $|Y|^2 \leqslant 2r - 2$. **Then $V_1 \cap B_r$ is the set of all pairs $(Y, X) \in (\mathbb{R}^n)^2$ with $X = \sqrt{1 + |Y|^2} X'$, where $|X'| = 1$, and $Y \in T_{X'}S^{n-1}$ a tangent vector such that $|Y| \leqslant \sqrt{2r - 2}$.** ■

Milnor fibration (3)

DEFINITION: Let $B_r \subset C$ be a disk of radius r centered in 0. In assumption of Claim 1, **the Milnor fibration of the quadratic form q** is the fibration $q^{-1}(B_r \backslash 0) \cap B_\varepsilon \longrightarrow B_r \backslash 0$. **It is locally trivial when $0 < r < \varepsilon$, as follows from Claim 1.** Its fibers are homotopy equivalent to a sphere S^{n-1} (Claim 1).

PROPOSITION: The monodromy around zero **multiplies the generator of $H^{n-1}(q^{-1}(t) \cap B_\varepsilon)$ by -1 if n is odd, and by 1 if n is even.**

Proof: Consider a circle $\delta e^{\sqrt{-1}\,t} \subset B_r$. The monodromy takes $(z_1, ..., z_n) \in V_\varepsilon$ to $(e^{\sqrt{-1}\,t/2} z_1, ..., e^{\sqrt{-1}\,t/2} z_n)$. In this formula, we divide by 2 because $q(e^{\sqrt{-1}\,t/2} z_1, ..., e^{\sqrt{-1}\,t/2} z_n) = e^{\sqrt{-1}\,t} q(z_1, ..., z_n)$. **The rotation by 2π takes $(z_1, ..., z_n)$ to $(-z_1, ..., -z_n)$.** The corresponding map $x \mapsto -x$ changes the orientation of the sphere when $n-1$ is even, and preserves it when it is odd.

■

Cohomology of the unit sphere bundle

Let $M := V_1 \cap B_r$ be the fiber of the Milnor fibration, considered as a manifold with boundary. Then ∂M is a unit sphere bundle STS^{n-1} in TS^{n-1}. Its cohomology are computed as follows

CLAIM: When $n = 3$, STS^{n-1} is $\mathbb{R}P^3$, and its rational cohomology are the same as for a sphere. **When $n > 3$, one has $H^*(STS^{n-1}) = H^*(S^{n-1} \times S^{n-2})$.**

REMARK: It is easy to see that STS^{n-1} has a cellular decomposition with 4 cells in dimensions $0, n-1, n-2, 2n-3$. **We need to show that the boundary map from the $n-1$-cell to $n-2$-cell induces zero in homology.**

Proof. Step 1: Let $X \longrightarrow Y$ be a holomorphic $\mathbb{C}P^1$-bundle on a Kähler manifold, and $Y_1 \subset X$ its section. **Gysin isomorphism gives $H^{n+2}(X, Y_1) = H^n(Y_1)$,** with the isomorphism provided by the multiplication by the Thom class.

Cohomology of the unit sphere bundle (3)

Step 2: Let $Q\subset \mathbb{C}P^n$ be the smooth quadric, and $C(Q)$ its cone. As we have already shown, STS^{n-1} is the boundary of the unit ball intersected with $C(Q)$. Blowing up the origin of $C(Q)$, we obtain the total space $\operatorname{Tot}(L)$ of an anti-ample line bundle L on Q. Let $X:=\mathbb{P}(L\oplus \mathcal{O}_Q)$. The natural map $X\stackrel{\pi}{\longrightarrow} Q$ is a $\mathbb{C}P^1$-bundle. Let $Y_1\subset X$ be the section given by $(z,0)\subset X$; its complement is isomorphic to $\operatorname{Tot}(L)$. Gysin isomorphism (Step 1) gives $H^{n+2}(X,Y_1)=H^n(Q)$. However, the intersection of $X\backslash Y_1$ and a neighbourhood of Y_1 is homotopy equivalent to the boundary of the sphere bundle in $\operatorname{Tot}(L)$, which brings the long exact sequence of a pair

$$\begin{aligned}... \longrightarrow H^{k-1}(STS^{n-1}) \longrightarrow H^k(\operatorname{Tot}(L), STS^{n-1}) \longrightarrow \\ \longrightarrow H^k(\operatorname{Tot}(L)) \longrightarrow H^k(STS^{n-1}) \longrightarrow H^{k+1}(\operatorname{Tot}(L), STS^{n-1}) \longrightarrow ... \quad (*)\end{aligned}$$

Step 3: Rewriting $H^k(\operatorname{Tot}(L))=H^k(Q)$ (these spaces are homotopy equivalent) and using $H^k(\operatorname{Tot}(L), STS^{n-1})=H^{k-2}(Q)$ (Gysin isomorphism), the exact sequence (*) becomes

$$\begin{aligned}... \longrightarrow H^{k-1}(STS^{n-1}) \longrightarrow H^{k-2}(Q) \longrightarrow \\ H^k(Q) \longrightarrow H^k(STS^{n-1}) \longrightarrow H^{k-1}(Q) \longrightarrow ... \quad (**)\end{aligned}$$

Cohomology of the unit sphere bundle (2)

Step 3 (2): We have rewritten (*) to obtain

$$... \longrightarrow H^{k-1}(STS^{n-1}) \longrightarrow H^{k-2}(Q) \longrightarrow \\ H^k(Q) \longrightarrow H^k(STS^{n-1}) \longrightarrow H^{k-1}(Q) \longrightarrow ... \quad (**)$$

The cohomology of the quadric is $\mathbb{Z}$ in all even dimensions except the middle, when it is $\mathbb{Z} \oplus \mathbb{Z}$, which is easy to see from the Lefschetz hyperplane section theorem. The cohomology of STS^{n-1} is zero outside of dimensions $0, n-1, n-2, 2n-3$, because it has cellular decomposition which has 1 cell in each of these dimensions. This implies that the map $H^{k-2}(Q) \longrightarrow H^k(Q)$ of (**) is non-zero for low k. Since this map is multiplicative, it is given by multiplication with the hyperplane section, hence it is non-zero for all k. This leaves the extra generator of middle dimension $n-1$, which appears twice in this exact sequence, once with $H^{k-2}(Q)$ and once with $H^k(Q)$. In the first case it implies that $\operatorname{rk} H^{n-2}(STS^{n-1}) = 1$, in the second case that $\operatorname{rk} H^{n-1}(STS^{n-1}) = 1$. ■

Milnor fibration: cohomology of a pair

Since M is the ball bundle in TS^{n-1}, it is homotopy equivalent to S^{n-1}. Its boundary is STS^{n-1}, and its cohomology we have just computed. Then, **the exact sequence of the pair gives**

$$H^{n-2}(M) = 0 \longrightarrow H^{n-2}(\partial M) = \mathbb{R} \longrightarrow H^{n-1}(M, \partial M) \longrightarrow$$
$$\longrightarrow H^{n-1}(M) = \mathbb{R} \longrightarrow H^{n-1}(\partial M) = \mathbb{R} \longrightarrow H^n(M, \partial M) \longrightarrow H^n(M) = 0,$$

which implies that $H^{n-1}(M, \partial M) = \mathbb{R}$ and $H^i(M, \partial M) = 0$ for all $i \neq n-1$.

"Picard-Lefschetz theory" deals with how the monodromy of the Gauss-Manin connection acts on these rank 1 cohomology groups; the precise answer is given by **Picard-Lefschetz formula**.

Lefschetz pencils (reminder from Lecture 8)

REMARK: Let $f \in \mathcal{O}_M$ be a holomorphic function on a complex manifold, and $x \in M$ its critical point. **The Hessian** is the matrix of second derivatives of f.

EXERCISE: Prove that **the Hessian form** $\operatorname{Hess}(f) \in \operatorname{Sym}^2(T_x^* M)$ **is independent from the choice of coordinates.**

DEFINITION: Let $X \subset M$ be a hypersurface locally given by an equation $f = 0$, for a holomorphic function $f \in \mathcal{O}_M$, and $x \in X$ its singular point. Then $df\Big|_{T_x M} = 0$. We say that x is **an ordinary double point of X** if $\operatorname{Hess}(f)$ is a non-degenerate 2-form on $T_x M$.

DEFINITION: Let L be a holomorphic line bundle on a compact complex manifold M, and $W \subset H^0(M, L)$ a 2-dimensional subspace. The zero sets of $x \in W \backslash 0$ is a collection D_t of divisors parametrized by $\mathbb{P}H^0(M, L) = \mathbb{C}P^1$. It is called **a pencil of divisors**. Its **base locus** is the intersection of all $D_t, t \in \mathbb{C}P^1$.

DEFINITION: **A Lefschetz pencil** is a pencil of divisors such that its base locus B is smooth and all D_t are smooth along B, and all singularities of D_t are ordinary double points.

Vanishing sphere and Lefschetz thimble of a quadratic form

DEFINITION: In assumption of Claim 1, let $S_\varepsilon \subset V_\varepsilon$ be the set $(z_1, ..., z_n) \in V_\varepsilon$ with each z_i real. Clearly, S_ε is a sphere of radius $\sqrt{\varepsilon}$. The group $\mathbb{C}^*$ acts on the set of all V_t by rotating the coordinates; we set S_t as the image of S_1 under this map, that is, as the set $(z_1, ..., z_n) \in V_t$ such that $(\sqrt{t}^{-1} z_1, ..., \sqrt{t}^{-1} z_n)$ is real. This set is called **the vanishing sphere** of the quadratic form q. The vanishing sphere is an $n-1$-dimensional sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$, hence it is a boundary of a closed n-dimensional ball e_t in B_ε. We call this ball **the Lefschetz thimble**.

REMARK: The vanishing sphere of V_t **is the set of minima for the function $x \mapsto |x|$ on V_t.**

Claim 2: The embeddings $e_t \hookrightarrow e_t \cup (V_t \cap B_\varepsilon) \hookrightarrow B_\varepsilon$ **are deformation retracts.**

Proof: These spaces are CW-complexes, and each embedding is a homotopy equivalence, because $V_t \cap B_\varepsilon$ is homotopy equvalent to a sphere which bounds e_t. A closed embedding of CW-complexes inducing homotopy equivalence **is a deformation retract (prove this as an exercise) (also find a direct proof of homotopy equivalence).** ■

Lefschetz thimbles for a fibration

REMARK: Let $f: \mathbb{C}^n \longrightarrow \mathbb{C}$ be a function with isolated critical point p. Assume that Hess(f) is non-degenerate in p. **By a holomorphic version of Morse lemma, in some holomorphic coordinates $z_1, ..., z_n$ around p, one has $f(z_1, ..., z_n) = f(p) + \sum z_i^2$.**

DEFINITION: Let $f: X \longrightarrow D$ be a proper holomorphic map of a manifold to a disk. Assume that this projection is a submersion outside of a point $x \in X$, such that $f(x) = 0$, and assume that Hess(f) is non-degenerate in x. Then $f(z_1, ..., z_n) = \sum z_i^2$ in a sufficiently small neighbourhood U of x. Take a non-zero $t \in D$. For a sufficiently small $|t|$, the vanishing sphere of $f^{-1}(t)$ and its Lefschetz thimble belongs to U. We call these sets **the vanishing sphere** and **the Lefschetz thimble** of the singularity f.

PROPOSITION: Let $f: X \longrightarrow D$ be a proper holomorphic map of a manifold to a disk. Assume that this projection is a submersion outside of a point $x \in X$, and assume that Hess(f) is non-degenerate in x. Let $e_t \subset X$ be the Lefschetz thimble, defined in a small neighbourhood of x, with the boundary on $X_t := f^{-1}(t)$, where $t \in D$ is a point in a small neighbourhood of $f(x)$. **Then the natural map $e_t \cup X_t \hookrightarrow X$ is a deformation retract.**

Proof: Next slide.

Lefschetz thimbles for a fibration (2)

Proof. Step 1: We can always pass to a small neighbourhood U of $x_0 := f(x)$, and replace X with $f^{-1}(U)$; this manifold is homotopy equivalent to X by Ehresmann theorem (indeed, it is a deformation retract of X; **prove it**). Removing an open submanifold with a boundary which is everywhere transversal to $f^{-1}(t)$, we obtain a manifold $X_B \subset X$ with a boundary; the fibration $X_B \longrightarrow D$ is locally trivial by Ehresmann theorem for compact manifolds with boundary. Then X is obtained by gluing X_B and the interior of B. Shrinkind D if necessary, may assume that **the interior of B is identified with an open subset $B \subset \mathbb{C}^n$ such that the map f restricted to B has a single Morse singularity in 0, and its fibers are transversal to the boundary of B.**

Step 2: Let $X' := X \backslash B$. By Ehresmann theorem again, $X' \cap X_t$ is a deformational retract of X', where $X_t = f^{-1}(t)$. **Claim 2 implies that $X_t \cup e_t$ is a deformational retract of X.** Indeed, we the fibration $f|_{X'}$ is trivial, hence we can retract X' to $X' \cap X_t$ and at the same time using Claim 2 retract B to $(X_t \cap B) \cup e_t$, making sure these two retractions agree on the boundary. ∎

Blown-up Lefschetz pencil

COROLLARY: Let L be a Lefschetz pencil on a projective manifold X, and A its base locus. Denote by $\tilde{X}$ the blow-up of X in A. The Lefschetz pencil defines a holomorphic map $\pi: \tilde{X} \longrightarrow \mathbb{C}P^1$, and the divisors D_t of the Lefschetz pencil are identified with the fibers of π. Assume that D_∞ is smooth, and let $D_0 \subset \tilde{X}$ be a divisor which is not necessarily smooth. **Then $\tilde{X}\backslash D_\infty$ is homotopy equivalent to D_0 with n-dimensional balls ("Lefschetz thimbles") glued to all its vanishing spheres.**

Proof: All singularities of the map $\tilde{X}\backslash D_\infty \longrightarrow \mathbb{C}$ have only ordinary double points. By the previous proposition, there exists a deformational retraction of $\tilde{X}\backslash D_\infty$ to X_0 with Lefschetz thimbles glued to all its vanishing spheres. ∎

REMARK: This result **can be used to prove the Lefschetz hyperplane section theorem.**

Generalized Dehn twist

DEFINITION: A **support** of a diffeomorphism $A: M \longrightarrow M$ is the set of al $m \in M$ such that $A \neq \mathrm{Id}$ in any neighbourhood of m. A diffeomorphism is **compactly supported** if its support is compact.

DEFINITION: We identify points of TS^n with pairs $X, Y \in \mathbb{R}^n$, such that $|X| = 1$ and $Y \perp X$. Choose a smooth function $\theta: \mathbb{R}^{\geqslant 0} \longrightarrow [0,\pi]$ such that $\theta(t) = 0$ when t is small, and $\theta(t) = \pi$ when it is large. A **generalized Dehn twist** is a compactly supported diffeomorphism of TS^n which takes (X, Y) to $R_\theta(X,Y)(-X,-Y)$, where $R_\theta(X,Y)$ is the rotation by angle θ in the plane $\langle X, Y\rangle$.

REMARK: Another interpretation of generalized Dehn's twist implies that it is actually a symplectomorphism: using the standard metric, we identify TS^n and T^*S^n. Consider the geodesic flow on $TS^n = T^*S^n$; it is a Hamiltonian flow associated with the Hamiltonian $h(v) = |v|^2$, hence it is a symplectomorphism. It acts as -1 the sphere of radius π. **Glue together the composition of $(X,Y) \mapsto (X,-Y)$ and the geodesic flow on $\{(X,Y) \ \mid\ |Y| \leqslant \pi\}$ and the identity map on $\{(X,Y) \ \mid\ |Y| \geqslant \pi\}$.** This will give a symplectomorphism, which is homotopy equivalent to the generalized Dehn twist defined above **(do this as an exercise).**

Generalized Dehn twist and Lefschetz pencils

CLAIM: Consider the map $q: \mathbb{C}^n \longrightarrow \mathbb{C}$, with $q(z_1, ..., z_n) = \sum_i z_i^2$, and let $V_t := q^{-1}(t)$ be its fiber, where $t \in \mathbb{C}$ runs through the circle $|t| = 1$. Using the diffeomorphism $V_t \longrightarrow TS^{n-1}$ and trivializing this fibration for $|Y| > 1$ as above, we obtain that the monodromy of V_t is given by a compactly supported diffeomorphism D. **Then D is homotopic to the generalized Dehn twist.**

Proof: On the vanishing sphere, D acts as $x \longrightarrow -x$; for general point $(X, Y) \in V_t$, we interpret X as the real part of the complex coordinate, Y its imaginary part, hence the Dehn twist is a complex rotation by θ on the corresponding plane. This map takes $x \in V_t$ to $-\theta x \in V_{\theta^2 t}$. ■

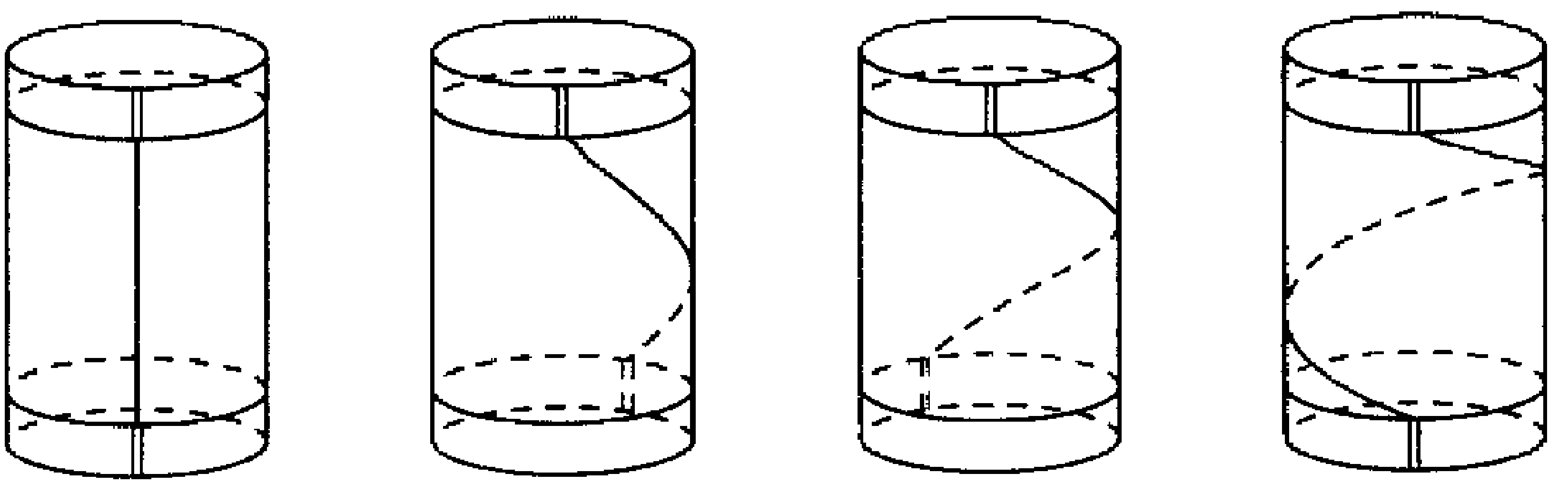
The variation map

Let (X, Y) be pair of CW-complexes, and (X_t, Y_t), $t \in S^1$ a locally trivial fibration over a circle. Assume that the fibration Y_t is trivial, and fix its trivialization. We choose a local trivialization of (X_t, Y_t) which is trivial on $Y_t = Y_0$. **This is all possible, for example, when $(X_t, Y_t) \longrightarrow S^1$ is a smooth submersion, and Y_t are submanifolds or a boundary of X_t, existence of such trivialization in this case follows from the Ehresmann theorem (prove it)**.

REMARK: Homology cycles of the pair (X, Y) **are singular (or simplicial, or cellular) cycles which have boundary in Y.**

The variation map (2)

DEFINITION: Assume that Y_t has a neighbourhood W_t in X_t which can be retracted to Y_t, and this retraction is continuous in t. Then all W_t, $t \in [0,1]$ can be identified. Taking a cycle C_0 representing a class in $H_d(X_0, Y_0)$, we move this class along with t keeping $C_t \cap W_t$ the same; the end cycle C_1 is equal to C_0 in W_0, hence the difference $C_0 - C_1$ defines a homology class in $H_d(X_0 \backslash Y_0)$.



The variation map for the Dehn twist

Thus constructed map $H_d(X_0, Y_0) \longrightarrow H_d(X_0 \backslash Y_0)$ is called **the variation map**.

Picard-Lefschetz formula

THEOREM: (Picard-Lefschetz formula) Let $q: X \longrightarrow D$ be a proper, holomorphic map to a disk, with a single ordinary double critical point over 0, $\dim X = n+1$, Mon the monodromy of the Gauss-Manin local system, considered as a automorphism of its cohomology, L the vanishing sphere, $l \in H^n(M, \mathbb{Z})$ its cohomology class, and $v \in H^n(M, \mathbb{Z})$ any other cycle. **Then** $\operatorname{Mon}(v) = v + (v,l)l$**, where** (v,l) **denotes the intersection form.**

Proof: The statement would follow if we prove that $\operatorname{Var}(v) = (v,l)l$ locally around the vanishing sphere. **In dimension 1, this is clear from the picture of the Dehn twist above.** Generally, it would follow if we prove that $D: TS^n \longrightarrow TS^n$ takes a fiber of the tangent bundle to a subvariety intersecting the general fiber once. However, the exponential map on the sphere is bijective from a ball of radius π in the fiber to the sphere without the diametral opposite point, hence the image of the fiber intersects the generator of $H_n(M_t, \partial M_t)$ precisely once. ■

COROLLARY: Consider a family $q: X \longrightarrow D$ of varieties defined as above. When its fibers are even-dimensional, the monodromy **acts in cohomology by reflections centered in the vanishing sphere, which has square** -2**.** For odd-dimensional manifolds, **the monodromy has infinite order.** ■