Variations of Hodge structures

lecture 10: Proof of Noether-Lefschetz theorem

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Fixed part theorem (reminder)

THEOREM: (Deligne-Griffiths-Schmid's fixed part theorem) Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base, and b a parallel section of B. **Then all (p,q)-components of b are also parallel.**

Proof: Lecture 4. ■

REMARK: This statement **also holds when M is quasiprojective (we prove it later in this course, if time permits).**

COROLLARY: Let $(B, \nabla, B = \bigoplus_{p+q=w} B^{p,q})$ be a variation of polarized Hodge structures over a compact base. Assume that the monodromy of ∇ is trivial. **Then the Hodge decomposition is preserved by ∇**, that is, **the corresponding variation of Hodge structures is constant.**

Proof: Consider a basis $e_1, ..., e_n$ of $B|_x$ such that each e_i belongs to some $B^{p,q}$. Since the monodromy of ∇ is trivial, we can extend each e_i to a parallel section $\tilde{e}_i$ of B. The Hodge components of each of $\tilde{e}_i$ are parallel, but it has only one Hodge component at x. **Therefore, $\tilde{e}_i$ has only one Hodge component at each point of M,** and the corresponding Hodge decomposition is also constant. ■

Deligne's semisimplicity theorem (reminder)

DEFINITION: We further weaken the notion of real Hodge structures, defining **a complex Hodge structure on a vector space A**, which is a decomposition $A = \bigoplus A^{p,q}$, without the assumption $\overline{A^{p,q}} = A^{q,p}$. **A complex VHS** is a decomposition $V = \bigoplus V^{p,q}$ of a flat vector bundle (V, ∇) such that ∇ acts on $V = \bigoplus V^{p,q}$ satisfying the Griffiths transversality.

THEOREM: (Deligne's semisimplicity theorem)
Let $(V, \nabla, V = \bigoplus V^{p,q})$ be an integer, polarized VHS over a compact (or quasiprojective) base. **Then the flat bundle (V, ∇) can be decomposed as $V = \oplus_i L_i \otimes_{\mathbb{C}} W_i$, where L_i are flat bundles with irreducible monodromy, and W_i complex vector spaces.** Moreover, **each L_i is equipped with a structure of a complex VHS, and each W_i with a complex Hodge structure,** in such a way that this decomposition is compatible with the Hodge structures.

Proof: Lecture 5.

REMARK: This decomposition **is not necessarily compatible with the integer (or even rational) structure** on V.

Noether-Lefschetz theorem (reminder from Lecture 7)

DEFINITION: Let M be a compact Kähler manifold. Denote the lattice $H^{1,1}(M) \cap H^2(M, \mathbb{Z})$ by $NS(M)$. This lattice is called **Picard lattice**, or **Neron-Severi lattice**. The number $\operatorname{rk} NS(M)$ is called **the Picard rank** of M.

THEOREM: Let X be a general hypersurface of degree $d \geqslant 4$ in $\mathbb{C}P^n$, with $n = 3$. **Then its Picard rank is 1.**
Proof: Later today.

REMARK: This statement is false for $d = 3$, $n = 3$. Indeed, a smooth cubic surface in $\mathbb{C}P^3$ is $\mathbb{C}P^2$ blown up in 6 points (Clebsch), which gives $\operatorname{rk} NS(X) = 7$.

REMARK: This statement is false for $d = 2$, $n = 3$. Indeed, a smooth quadric in $\mathbb{C}P^3$ is $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Irreducibility of the monodromy (reminder from Lecture 7)

Let $S = H^0(\mathbb{C}P^3, \mathcal{O}(d))$ be the space of all homogeneous polynomials of 4 variables of degree d, and $S_0 \subset S$ an open subset which corresponds to polynomials which give smooth surfaces of degree d in $\mathbb{C}P^3$. Consider the incidence variety $Z \subset \mathbb{C}P^3 \times \mathbb{P}S$ formed by all pairs

$$\{(z, f) \in \mathbb{C}P^3 \times \mathbb{P}S \quad f(z) = 0\}$$

Clearly, Z is fibered over $\mathbb{P}S$ with the fibers at $f \in \mathbb{P}S$ isomorphic to the corresponding surface of degree d. Let Z_0 be the preimage of S_0 in Z. By construction, **the fibration $Z_0 \longrightarrow S_0$ is a smooth, proper submersion with projective fibers.** Recall that "primitive part" of $H^2(S)$, for a projective complex surface, is the orthogonal complement to the Kähler class, taken with respect to the intersection form; this is the space equipped with the polarized Hodge structure.

THEOREM: The monodromy of the Gauss-Manin connection associated with the primitive second cohomology of the fibers of $Z_0 \longrightarrow S_0$ is irreducible.

Proof: Later today. **The proof uses Lefschetz pencils, vanishing cycles and Picard-Lefschetz theory.**

Noether-Lefschetz theorem and irreducibility of monodromy (reminder)

THEOREM: Let X be a general hypersurface of degree $d \geqslant 4$ in $\mathbb{C}P^n$, with $n = 3$. **Then its Picard rank is 1.**

Proof. Step 1: Let Y be a general degree d surface, $H^2(Y)_{\text{prim}}$ its primitive second cohomology, $H^2(Y)_{\text{prim}} := \{\eta \in H^2(Y) \mid \eta \wedge \omega = 0\}$, and $W \subset \mathbb{P}H^0(\mathbb{C}P^3, \mathcal{O}(d))$ the set of all points which correspond to smooth degree d surfaces. Let V be the Gauss-Manin local system with the fiber $H^2(Y)_{\text{prim}}$ over W associated with this fibration, and $V_0 \subset V$ space of all elements which remain of type (1,1) if parallel transported over all W. Then $V_0 \subset V$ is a local subsystem. **Since V is irreducible, this subsystem is empty (and in this case the Noether-Lefschetz loci have positive codimension), or it is everything.**

Step 2: In the second case, $H^{2,0}(Y) = 0$. However, the adjunction formula gives $K_Y = K\mathbb{C}P^2 \otimes N_Y = \mathcal{O}(-4) \otimes O(d) = \mathcal{O}(d-4)$, **and it has sections when $d \geqslant 4$.** ■

Discriminant variety

REMARK: Clearly, $H^0(\mathbb{C}P^n, \mathcal{O}(d))$ is the space of degree d polynomial functions on $\mathbb{C}^{n+1}$. Therefore, **any point $x \in \mathbb{P}H^0(\mathbb{C}P^n, \mathcal{O}(d))$ corresponds to a degree d hypersurface, and the space $\mathbb{P}H^0(\mathbb{C}P^n, \mathcal{O}(d))$ parametrizes them all.**

DEFINITION: Let $D \subset \mathbb{P}H^0(\mathbb{C}P^n, \mathcal{O}(d))$ be the set of all points which correspond to singular hypersurfaces. This set is called **the discriminant variety**.

CLAIM: The discriminant variety is irreducible.

Proof: Let V be the image of the Veronese embedding $\mathbb{C}P^n \longrightarrow \mathbb{P}H^0(\mathbb{C}P^n, \mathcal{O}(d))^*$. Then $D = V^\vee$. As shown in Lecture 9, $(V^\vee)^\vee = V$. If $V^\vee$ were not irreducible, V would be non-irreducible as well. ■

CLAIM: The discriminant variety is a divisor.

Proof: Let $\mathbb{C}P^n \xrightarrow{\nu} \mathbb{P}H^0(\mathbb{C}P^n, \mathcal{O}(d))^*$ be the Veronese embedding. Then D is the set of all hyperplane sections which are tangent to $V = \operatorname{im} \nu$ at some point, that is, the projective dual $V^\vee$. It remains to show that $V^\vee$ is a hypersurface; **we leave this as an exercise (discuss).** ■

Fundamental groups of the complement

THEOREM: (Zariski) Let $Z \subset \mathbb{C}P^n$ be a subvariety, and $U := \mathbb{C}P^n \backslash Z$ its complement. Consider a line $L \subset \mathbb{C}P^n$ which intersects Z transversally. **Then the natural map $\pi_1(L \cap U) \longrightarrow \pi_1(U)$ is surjective.**

Proof. Step 1: Let $A \subset B$ be a positive-dimensional complex subvariety in a complex manifold. **Clearly, $\pi_1(B \backslash A) \longrightarrow \pi_1(B)$ is surjective (prove this as an exercise).**
Step 2: Let $p \in U$, and $U_0 \subset U$ the space of all points $x \in U \backslash p$ such that the line passing through p and x is transversal to Z. Let $\mathcal{L}$ be the space of all lines passing through p and transversal to Z. Clearly, the natural map $U_0 \longrightarrow \mathcal{L}$ **is a Serre fibration with a fiber over $l \in \mathcal{L}$ identified with $l \backslash (p \cup (l \cap Z))$.**

Step 3: This gives an exact sequence $\pi_1(\mathbb{C} \backslash l \cap Z) \longrightarrow \pi_1(U_0) \xrightarrow{\pi} \pi_1(\mathcal{L}) \longrightarrow 0$. Let $B \subset U$ be a ball with center in p. For any loop $\gamma : S^1 \longrightarrow \mathcal{L}$, consider the corresponding punctured disk bundle $\mathcal{D}_\gamma$ over S^1, with fiber $(B \cap \gamma(t)) \backslash \{p\}$ at any $t \in S^1$. Since $\mathcal{D}_\gamma$ is homotopy equivalent to a torus, it always admits a section $\sigma_\gamma : S^1 \longrightarrow B \backslash \{p\}$. **Using this section, we obtain a section $\sigma : \pi_1(\mathcal{L}) \longrightarrow \pi_1(U_0)$, taking a path γ to the path $\sigma_\gamma : S^1 \longrightarrow U_0$.**

Step 4: Consider now the natural map $\pi_1(U_0) \longrightarrow \pi_1(U)$ induced by the open embedding. **This map vanishes on the image of σ, hence $\pi_1(\mathbb{C} \backslash l \cap Z)$ surjects onto the image of $\pi_1(U_0)$ in $\pi_1(U)$.** ■

Lefschetz pencils (reminder from Lecture 8)

REMARK: Let $f \in \mathcal{O}_M$ be a holomorphic function on a complex manifold, and $x \in M$ its critical point. **The Hessian** is the matrix of second derivatives of f.

DEFINITION: Let $X \subset M$ be a hypersurface locally given by an equation $f = 0$, for a holomorphic function $f \in \mathcal{O}_M$, and $x \in X$ its singular point. Then $df\Big|_{T_xM} = 0$. We say that x is **an ordinary double point of X** if $\operatorname{Hess}(f)$ is a non-degenerate 2-form on T_xM.

DEFINITION: Let L be a holomorphic line bundle on a compact complex manifold M, and $W \subset H^0(M, L)$ a 2-dimensional subspace. The zero sets of $x \in W \backslash 0$ is a collection D_t of divisors parametrized by $\mathbb{P}H^0(M, L) = \mathbb{C}P^1$. It is called **a pencil of divisors**. Its **base locus** is the intersection of all $D_t, t \in \mathbb{C}P^1$.

DEFINITION: A Lefschetz pencil is a pencil of divisors such that its base locus B is smooth and all D_t are smooth along B, and all singularities of D_t are ordinary double points.

Projective dual variety (reminder from Lecture 8)

Let $\mathbb{P}^n = \mathbb{P}V$ be a complex projective space. We denote $\mathbb{P}V^*$ by $\check{\mathbb{P}}^n$, and **identify it with the set of all projective hyperplanes in $\mathbb{P}^n$.**

DEFINITION: Let $X \subset \mathbb{P}^n$ be a complex manifold. **The projectively dual variety** $X^\vee$ is the space of all hyperplanes $V \in \check{\mathbb{P}}^n$ tangent to some point in X.

Let $A = \mathbb{C}P^{n-2} \subset \mathbb{P}^n$ be a projective subspace, embedded in the standard way, and $Z \subset \check{\mathbb{P}}^n$ the set of all planes containing A. Clearly, Z is a pencil of divisors. **We identify Z with its image $L = \mathbb{C}P^1 \subset \check{\mathbb{P}}^n$.**

Fix a closed, irreducible, smooth $X \subset \mathbb{P}^n$. **The intersection of Z with X gives a pencil of divisors on X with the base set $Y = A \cap X$.**

DEFINITION: The pencil $L \subset \check{\mathbb{P}}^n$ **is transversal to $X^\vee$** if L meets $X^\vee$ only in its smooth part, and is transversal to $X^\vee$ in all intersection points.

CLAIM: A general line $L \subset \check{\mathbb{P}}^n$ is transversal to $X^\vee$.

THEOREM: Let $X \subset \mathbb{P}^n = \mathbb{P}^n V$ be a submanifold, and $L \subset \check{\mathbb{P}}^n = \mathbb{P}^n V^*$ a line which is transversal to $X^\vee$. **Then L defines a Lefschetz pencil on X.**

A form of Lefschetz hyperplane section theorem

Claim 1: Let $Y \subset X$ be a smooth hyperplane section of a projective manifold, $\dim_{\mathbb{C}} M$. **Then the natural map $H_k(Y) \longrightarrow H_k(X)$ is injective for all $k \neq n$.**

Proof: It is an isomorphism for $k < n$ by Lefschetz hyperplane section theorem. For $k > n$, we consider the Poincaré dual map $H^{2n-k}(Y) \longrightarrow H^{2n-k+2}(X)$. Its composition with the restriction $H^{2n-k+2}(X) \longrightarrow H^{2n-k+2}(Y)$ is dual to the multiplication $\cap H: H^{2n-k}(Y) \longrightarrow H^{2n-k+2}(Y)$ with the hyperplane section, because it is dual to a map which takes a cycle α on Y, uses the intersection with this cycle to obtain a functional on the cohomology of X, and then restricts this functional to Y. This is the same as take a cycle on Y, move it to X, and intersect it with α, which is the same as to intersect it with $\alpha \cap H$. Since the map $\cap H: H^{2n-k}(Y) \longrightarrow H^{2n-k+2}(Y)$ is injective for all $2n - k < n$, the embedding $H_k(Y) \longrightarrow H_k(X)$ is injective for all $k > n$. ■

Vanishing cohomology

DEFINITION: Let $Y \subset X$ be a smooth hyperplane section of a projective manifold, $\dim_{\mathbb{C}} Y = n$. **The group of vanishing classes** $H_n(Y)_{\mathsf{van}}$, or **vanishing cohomology** is the kernel of the natural map $H_n(Y) \longrightarrow H_n(X)$, or the Poincare dual map $H^n(Y) \longrightarrow H^{n+2}(X)$; in the later case it is denoted by $H^n(Y)_{\mathsf{van}} \subset H^n(Y)$.

DEFINITION: Let (M, I, ω) be an n-dimensional Kähler manifold, and $k \leqslant n$. Consider the Lefschetz $SL(2)$-triple (L, H, Λ) acting on the cohomology of M. **Primitive part of cohomology** $H^*(M)_{\mathsf{prim}}$, or **the space of primitive classes** is the kernel of $H^k(M) \xrightarrow{\Lambda} H^{k-2}(M)$.

CLAIM: Let (X, ω) be a projective manifold, with the Kähler class ω homologous to the fundamental class $[H]$ of the hyperplane section. Then **all vanishing classes in $H^n(Y)$ are primitive: $H^n(Y)_{\mathsf{van}} \subset H^n(Y)_{\mathsf{prim}}$**

Proof: By definition, vanishing cohomology is the kernel of the natural map $H^n(Y) \longrightarrow H^{n+2}(X)$. The composition of this map and the restriction $H^{n+2}(X) \longrightarrow H^{n+2}(Y)$ is $\cap H: H^n(Y) \longrightarrow H^{n+2}(Y)$ (this was shown the proof of Claim 1), and the kernel of $H^n(Y) \xrightarrow{\cap H} H^{n+2}(Y)$ is primitive classes. ■

Vanishing cohomology for hypersurfaces

PROPOSITION: Let $Y \subset \mathbb{C}P^{n+1}$ be a general hypersurface of degree d, obtained as a hyperplane section of the image V of Veronese embedding, and L a Lefschetz pencil on V, such that Y is its fiber. **Then the vanishing cohomology coincide with the primitive cohomology.**

Proof: The restriction map $H^i(V) \longrightarrow H^i(Y)$ is injective for all $i \leqslant 2n$, because the cohomology of V are powers of the Kähler form, and the Kähler form is never exact. The vanishing cohomology is $\ker(H^n(Y) \longrightarrow H^{n+2}(V))$, and the primitive cohomology is the kernel of the composition of this map with the restriction $H^{n+2}(V) \longrightarrow H^{n+2}(Y)$; since the latter is injective, vanishing and primitive cohomology coincide. ∎

Vanishing cycles and Lefschetz pencils

THEOREM: Let L be a Lefschetz pencil on X, and $Y \subset X$ a smooth divisor in X. **Then the group $H_n(Y)_{\mathsf{van}}$ of vanishing cycles is generated by the homology classes of all vanishing spheres in L.**

Proof. Step 1: Let $\tilde X$ be the blowup of X in the base point set of L, $X_\infty \subset \tilde X$ a smooth fiber, and $\tilde X_\infty := \tilde X \backslash X_\infty$. Consider another smooth fiber $X_0 \subset \tilde X_\infty$. **Then $\tilde X_\infty$ is homotopy equivalent to X_0 with Lefschetz thimbles glued to all vanishing spheres** (Lecture 9). Therefore, the kernel of $H_n(Y) \longrightarrow H_n(X)$ contains the boundaries of the Lefschetz thimbles. Indeed, the space generated by vanishing spheres is the kernel of the natural map $H_n(Y) \longrightarrow \tilde H_n(\tilde X_\infty)$ and the embedding $Y \longrightarrow X$ is factorized through $Y \longrightarrow \tilde X_\infty$.

Step 2: It remains to show that the the map $H_n(\tilde X_\infty) \longrightarrow H_n(X)$ is injective; indeed, the map $Y \longrightarrow X$ is factorized through $Y \longrightarrow \tilde X_\infty$, and the kernel of the latter is $H^n(Y)_{\mathsf{van}}$ by Step 1. The long exact sequence of the pair $(\tilde X, \tilde X_\infty)$ together with the Thom isomorphism $H_n(\tilde X, \tilde X_\infty) \cong H_{n-2}(X_\infty)$ give the following

$$... \longrightarrow H_{n+1}(\tilde X) \longrightarrow H_{n-1}(X_\infty) \longrightarrow H_n(\tilde X_\infty) \longrightarrow H_n(\tilde X) \longrightarrow H_{n-2}(X_\infty) \longrightarrow ...$$

where $H_k(\tilde{X}) \longrightarrow H_{k-2}(X_\infty)$ is the homological version of the Gysin map, given by intersection with the fundamental class $[X_\infty]$. The Gysin map is surjective for $k \leqslant n+1$ by Lefschetz hyperplane section theorem, hence $H_n(\tilde{X}_\infty) \longrightarrow H_n(\tilde{X})$ is injective.

Step 3: To finish the proof, **it remains to show that the kernel of the natural map $H_n(\tilde{X}_\infty) \longrightarrow H_n(X)$ is the same as the kernel of $H_n(\tilde{X}_\infty) \longrightarrow H_n(\tilde{X})$.** This will follow if we prove that the map $H_n(\tilde{X}_\infty) \longrightarrow H_n(\tilde{X})$ is factorized through $H_n(X)$. However, all homology cycles on $\tilde{X}_\infty$ come from the cycles in the fibers of the projection to $\mathbb{C}P^1 \backslash \infty$; since all these fibers map injectively to X, the corresponding homology map is factorized through $H_n(X)$. ■

Vanishing cycles and Lefschetz pencils (2)

PROPOSITION: Let $X \subset \mathbb{C}P^m$ be a smooth projective manifold, and $X^\vee \subset \check{\mathbb{C}}P^m$ its projective dual, understood as the set of all non-transversal hyperplane sections. For each Lefschetz pencil $L \subset \check{\mathbb{C}}P^n$, let $S = \{x_1, ..., x_k\} \subset L = \mathbb{C}P^1$ be the set of points corresponding to the singular fibers $D_{x_1}, ..., D_{x_k}$, and $s_i \in D_{x_i}$ the singular points of D_{x_i}. Fix a smooth divisor D_x in the Lefschetz pencil, associated with $x \in \mathbb{C}P^1 \backslash S$, and let $v_1, ..., v_k \in H^n(D)$ be the vanishing cycles associated with $s_1, ..., s_k$. Consider the monodromy action of the group $\Gamma := \pi_1(\mathbb{C}P^1 \backslash S)$ on $H^n(D_x)$. **Then for each i, j, Γ contains an element u_{ij} such that the conjugate of v_i with u_{ij} is either v_j or $-v_j$.**

Proof: Next slide

REMARK: Note that **the classes v_i are not well-defined,** without specifying the path γ_i connecting x_i to x in $\mathbb{C}P^1 \backslash S$. However, **the different choices of γ_i are all conjugated by Γ-action** (Exercise 1), and the corresponding choices of v_i are conjugated by the monodromy of the local system.

Exercise 1: Let ξ be a free loop on a connected manifold M, and $p \in M$ a point. Let us connect $z \in \xi$ to p by a path γ. This gives an element of the fundamental group $\pi_1(p, M)$: a path $\gamma_\xi := \gamma \circ \xi \circ \gamma^{-1}$ which goes along γ to x, then around ξ and back to p along γ^{-1}. **Prove that all paths obtained this way are conjugated in $\pi_1(M)$.**

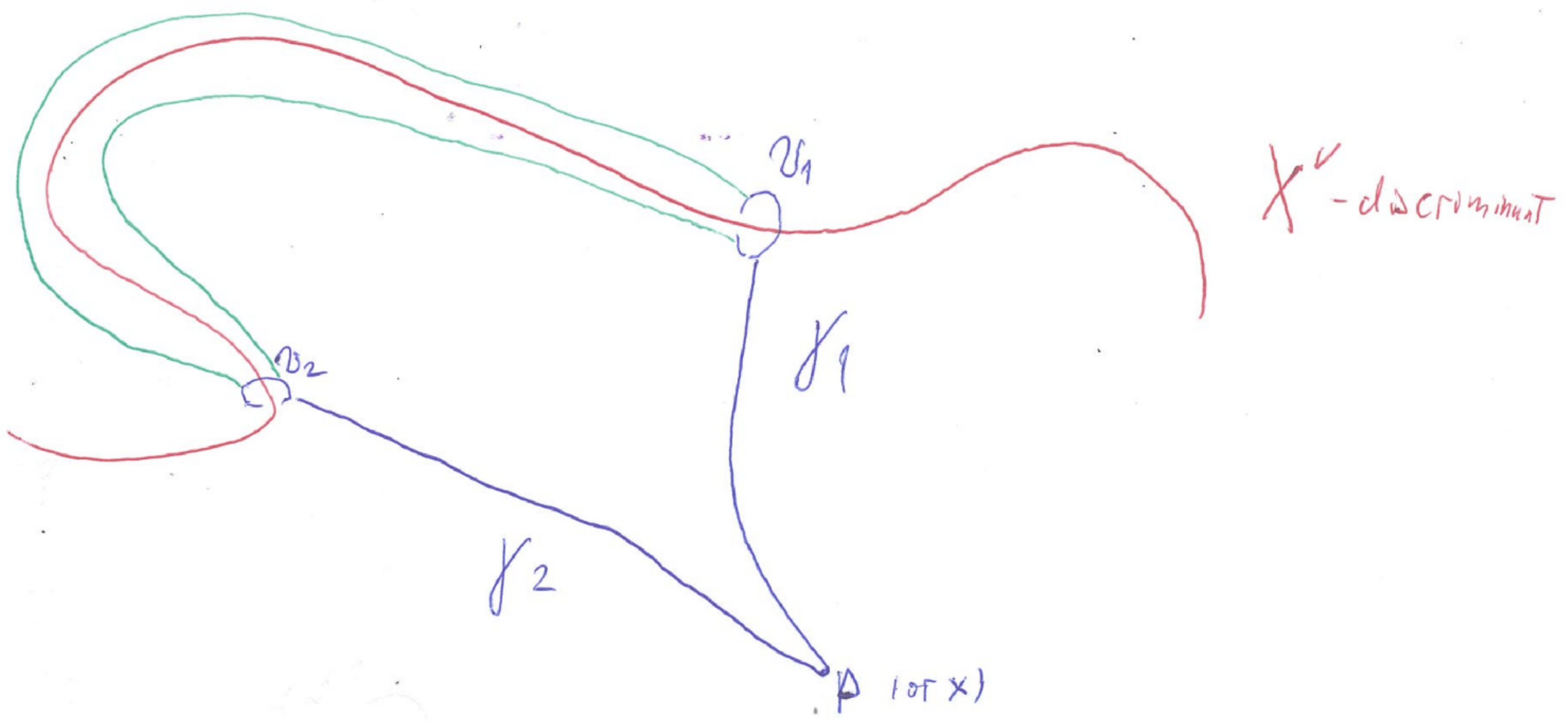
Vanishing cycles are permuted by the monodromy

PROPOSITION: Let $X \subset \mathbb{C}P^m$ be a smooth projective manifold, and $X^\vee \subset \check{\mathbb{C}}P^m$ its projective dual, understood as the set of all non-transversal hyperplane sections. For each Lefschetz pencil $L \subset \check{\mathbb{C}}P^n$, let $S = \{x_1, ..., x_k\} \subset L = \mathbb{C}P^1$ be the set of points corresponding to the singular fibers $D_{x_1}, ..., D_{x_k}$, and $s_i \in D_{x_i}$ the singular points of D_{x_i}. Fix a smooth divisor D_x in the Lefschetz pencil, associated with $x \in \mathbb{C}P^1 \backslash S$, and let $v_1, ..., v_k \in H^n(D)$ be the vanishing cycles associated with $s_1, ..., s_k$. Consider the monodromy action of the group $\Gamma := \pi_1(\mathbb{C}P^1 \backslash S)$ on $H^n(D_x)$. **Then for each i, j, Γ contains an element u_{ij} such that the conjugate of v_i with u_{ij} is either v_j or $-v_j$.**

Proof. Step 1: I would assume that n is even, to simplify the conventions. The proof for odd n is the same. For each v_i, consider the monodromy around the corresponding singular fiber. It acts on $H^n(D_x)$ as a reflection $x \mapsto x - 2\frac{(x,v_i)}{(v_i,v_i)}v_i$ around v_i. **Therefore, it would suffice to show that all these reflections are conjugate.** Also, the Γ-action on $H^n(D_x)$ is factorized through the $\pi_1(\check{\mathbb{C}}P^n \backslash X^\vee)$-action, hence it would suffice to show that these reflections are conjugate by elements of $\pi_1(\check{\mathbb{C}}P^n \backslash X^\vee)$.

Vanishing cycles are permuted by the monodromy (2)

Step 2: Consider the free loops ξ_i around x_i, and let γ_i connect these loops to x. As in the exercise above, these data give us elements $u_i := \gamma_i \circ \xi_i \circ \gamma_i^{-1}$ in $\pi_1(\check{\mathbb{C}}P^n \backslash X^\vee, x)$. The Gauss-Manin monodromy around u_i is the reflection around the corresponding vanishing cycle. Since $X^\vee$ is irreducible, the free loop ξ_i is homotopy equivalent to ξ_j or to ξ_j^{-1}.



Exercise 1 implies that the corresponding reflections are conjugate. ■

Monodromy representation on vanishing cycles is irreducible

The following corollary, applied to the image of the Veronese embedding, finishes the proof of Noether-Lefschetz theorem, which started in Lecture 7.

COROLLARY: Let $X \subset \mathbb{C}P^m$ be an $n+1$-dimensional projective manifold. Consider the set of all its transversal hyperplane sections, considered as a smooth fibration over $\mathbb{C}\check{P}^m \backslash X^\vee$. Let Y be a smooth hyperplane section, and $H^n(Y)_{\mathsf{van}}$ the vanishing cohomology. This defines the Gauss-Manin local system on $\mathbb{C}\check{P}^m \backslash X^\vee$ with the fiber $H^n(Y)_{\mathsf{van}}$. **Then the monodromy action of $\pi_1(\mathbb{C}\check{P}^m \backslash X^\vee)$ on $H^n(Y)_{\mathsf{van}}$ is irreducible.**

Proof: To simplify the conventions, we assume that n is even. We know that $H^n(Y)_{\mathsf{van}}$ is generated by vanishing cycles $v_1, ..., v_k$ of the Lefschetz pencils. As shown in Lecture 9, the monodromy group contains reflections $x \mapsto x - 2\frac{(x,v_i)}{(v_i,v_i)} v_i$, hence any sub-representation is either orthogonal to v_i or contains v_i. Since Γ acts transitively on the set of pairs $\{v_i, -v_i\}$, there are no sub-representations which are orthogonal to any given v_i. ■