RESEARCH OVERVIEW

MISHA VERBITSKY

Contents

1.1 Hodge theory on hyperkähler manifolds and its applications

Since early 1990-ies, I studied the applications of Hodge theory for topology of hyperkähler manifolds (the first paper, published in 1990, was written when I was 17 years old). I have shown that cohomology of a hyperkähler manifold admit an action of the Lie group $Sp(1,1)$, which is similar to Lefschetz' $SL(2)$-action. This was used to compute the cohomology algebra of a hyperkähler manifold, showing that its part generated by $H^2(M)$ is symmetric, up to the middle dimension. Among applications of these results, a proof of Mirror Conjecture for hyperkähler manifolds, and better understanding of hyperkähler subvarieties of hyperkähler manifolds and coherent sheaves.

This research culminated in the proof of global Torelli theorem, and discovering the ergodicity of the mapping class group action, which has many geometric consequences, including the proof of Kawamata-Morrison's cone conjecture (jointly with Ekaterina Amerik). Jointly with Ljudmila Kamenova, we used the same reasoning to prove many open conjectures about Kobayashi pseudometric on hyperkähler manifolds.

Research on deformation theory of rational curves on hyperkähler manifolds lead to development of a new notion, the MBM class, which serves the same role as the (-2)-class on a K3 surface. In addition to the proof of Kawamata-Morrison's cone conjecture, this lead to many new advances in the theory of automorphisms and bimeromorphic maps on hyperkähler manifolds. We proved that the centers of bimeromorphic contraction on a hyperkähler manifolds are deformation invariant within a divisor in the Teichmüller space of deformations, and used this observation to treat the shape of the Kähler cone of a hyperkähler manifold using the methods of hyperbolic geometry.

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1.2 Trianalytic subvarieties of hyperkähler manifolds and symplectic geometry

Since early 1990-ies, I studied trianalytic subvarieties in hyperkähler and hypercomplex manifolds. These are subvarieties which are complex analytic with respect to three complex structures I, J, K. It was shown that all complex subvarieties of a generic hyperkähler manifold are trianalytic. Also all deformations of trianalytic subvarieties are again trianalytic, and their deformation space is singular hyperkähler.

I have studied singularities of singular hyperkähler varieties, and shown that a normalization of such variety is smooth and hyperkähler. This applies to trianalytic subvarieties, which are examples of singular hyperkähler spaces.

These results were applied to Hilbert schemes of points on K3 and generalized Kummer varieties. For Hilbert schemes of points on K3, it was shown that its generic deformation has no subvarieties. For generalized Kummer varieties, a similar attempt (joint with D. Kaledin) was foiled by our imperfect understanding of birational geometry of holomorphic symplectic manifolds. Soon after publishing this paper, we found a counterexample to one of our statements.

The birational geometry of holomorphic symplectic manifolds was studied by D. Kaledin in several papers in much detail. Our joint work in this direction resulted in a proof of a deformation theorem, analoguous to Bogomolov-Tian-Todorov, for non-compact holomorphic symplectic manifolds. Going in the same direction, I have studied quotient singularities of holomorphic symplectic manifolds admitting a holomorphic symplectic resolution, and proved that such singularities are always quotients by groups generated by symplectic reflections.

Since then I published a series of paper dealing with complex and bimeromorphic geometry of Douady spaces of subvarieties in hyperkähler manifolds and in their twistor spaces. In particular, I have shown that the deformation spaces of complex curves in the twistor spaces of K3 surfaces are holomorphically convex (Stein for certain specific components). Also I have proven the "holography principle" for twistor spaces, showing that a meromorphic function on a twistor space of a small open subset of a hyperkähler manifold can be extended to the whole twistor space.

Research on trianalytic subvarieties also lead to curious development in symplectic geometry, joint with Jake Solomon; we have proven that the Fukaya category of a hyperkähler manifold is locally formal.

In another development related to symplectic geometry, we studied symplectic packing of hyperkähler manifolds by symplectic balls and ellipsoids. Using results about trianalytic subvarieties, we have proven that the symplectic packings of tori and hyperkähler manifolds are unobstructed; we also studied the ergodic properties of symplectic structures and used the ergodicity to prove new results about the symplectic packings by various other shapes, such as parallelograms and polydisks.

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1.3 Coherent sheaves on hyperkähler manifolds

In my second published paper I introduced the notion of hyperholomorphic bundles on hyperkähler manifolds. These are bundles with Hermitian connection with curvature of type (1,1) with respect to all complex structures induced by the hyperkähler structure.

It was shown that all stable bundles on generic hyperkähler manifolds admit a hyperholomorphic connection, which is unique. Conversely, every hyperholomorphic bundle is a direct sum of stable bundles.

The moduli spaces of such bundles wre shown to be singular hyperkähler, and the deformation unobstructed (except the first obstruction, known as Yoneda product).

Later, this notion was extended to coherent sheaves. Using results about trianalytic subvarieties, I have shown that a deformation of a hyperholomorphic bundle over a generic hyperkähler manifold M remains non-singular, unless M contains trianalytic subvarieties of complex codimension 2 (in the cases of a Hilbert scheme of K3 and the generalized Kummer variety, all trianalytic subvarieties have codimension > 2, except for 4-dimensional generalized Kummer).

If the deformation spaces of hyperholomorphic bundles are positive-dimensional (which is unknown yet), this should lead to new examples of hyperkähler manifolds.

In a joint paper with D. Kaledin, a non-Hermitian version of this notion was studied. We have shown that if the Hermitian assumption is dropped, the non-Hermitian hyperholomorphic connection on M becomes essentially the same as a holomorphic structure on the lifting of the corresponding bundle to the twistor space. In early 2000-ies this approach was used to study the category of coherent sheaves on generic K3 surfaces and tori. I have shown that this category is independent from the choice of a generic K3 or a torus of a given dimension.

Since then, I have written a few papers dealing with stable bundles on non-Kähler manifolds; this direction turned out to be quite fruitful. Jointly with Ruxandra Moraru, we have discovered that the moduli of stable bundles on a Hopf surface are generalized hyperkähler (in the sense of Hitchin and Gualtieri).

I have also discovered that in complex dimension > 2 any stable coherent sheaf on some elliptic fibrations (including the quasi-regular Vaisman manifolds, Calabi-Eckmann manifolds, and many others) are obtained as pullback from the leaf space, which is always a projective orbifold. This leads to many new results about the vector bundles and coherent sheaves on such manifolds. The notion of "positive elliptic fibration" was generalized to "positive toric fibration" in a paper dealing with complex geometry of compact homogeneous complex manifolds.

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1.4 Moduli spaces of framed instanton bundles on CP^3 and the rational curves on the twistor space

Jointly with D. Kaledin, I have constructed a correspondence between stable vector bundles on a twistor space of a hyperkaähler manifopld and rational curves in a twistor space of another hyperkähler manifold. This observation was used in early 2010-ies a collaboration with Marcus Jardim.

We have shown that the moduli space M of holomorphic vector bundles on CP^3 that are trivial along a line is isomorphic (as a complex manifold) to a subvariety in the moduli of rational curves of the twistor space of the moduli space of framed instantons on $\mathbb{C}^2$, called the space of twistor sections. This space admits an interesting geometric structure, called **a trisymplectic structure**.

A trisymplectic structure on a complex $2n$-manifold is a triple of holomorphic symplectic forms such that any linear combination of these forms has rank $2n$, n or 0. We have shown that a trisymplectic manifold is equipped with a holomorphic 3-web and the Chern connection of this 3-web is holomorphic, torsion-free, and preserves the three symplectic forms. Then we constructed a trisymplectic structure on the moduli of regular rational curves in the twistor space of a hyperkaehler manifold, and defined a trisymplectic reduction of a trisymplectic manifold, which is a complexified form of a hyperkaehler reduction. We proved that the trisymplectic reduction in the space of regular rational curves on the twistor space of a hyperkaehler manifold M is compatible with the hyperkaehler reduction on M.

As an application of these geometric ideas, we considered the ADHM construction of instantons. We have shown that the moduli space of rank r, charge c framed instanton bundles on CP^3 is a smooth, connected, trisymplectic manifold of complex dimension $4rc$. In particular, it follows that the moduli space of rank 2, charge c instanton bundles on CP^3 is a smooth complex manifold dimension $8c - 3$, thus settling a 30-year old conjecture of Barth and Hartshorne.

Trisymplectic structures, and their generalization, the k-symplectic structures, appear in many other domains in geometry, and seem to have many uses in addition to the study of instanton spaces.

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1.5 Hodge theory on manifolds with special holonomy and HKT-geometry

A **hypercomplex manifold** is a manifold with three complex structures I, J, K satisfying quaternionic relations. It is called **quaternionic Hermitian** if it has a quaternionic-invariant Riemannian structure. With each quaternionic Hermitian manifold (M, I, J, K, g), one can associate its canonical $(2, 0)$-form $\Omega = \omega_J + \sqrt{-1}\omega_K$. If this form is closed, M is called hyperkähler (this is one of possible definitions). If $\partial\Omega = 0$, (M, I, J, K, g) is called **an HKT-manifold** ("hyperkähler with torsion").

An HKT form is in many ways similar to a Kähler structure. One can define a potential, a version of Hodge theory, Kähler class and so on. In 2000-ies I studied hypercomplex geometry from this point of view. The Hodge theory (including Lefschetz-type $SL(2)$-action) was constructed for HKT manifolds with trivial canonical bundle. As an application, it was shown that a compact hypercomplex manifold which admits a Kähler metric also admits a hyperkähler structure. In another application (jointly with I. Dotti and M. L. Barbieris) it was shown that a hypercomplex nilmanifold admits an HKT structure if and only if it is abelian.

This result was also useful in hyperkähler geometry, where a strong vanishing result was shown, based on this Lefschetz -type $SL(2)$-action. It was shown that cohomology of a holomorphic line bundle L with $-c_1(L)$ outside of a dual Kähler cone vanish after the middle dimension. Moreover, form any holomorphic bunldle M, $H^i(B \otimes L^N) = 0$, for N sufficiently big, and $i > \frac{1}{2} \dim_{\mathbb{C}} M$.

The version of Hodge theory developed for the study of HKT manifolds, was also useful in other geometric sutuations, namely, for G_2-manifolds and nearly Kähler manifolds, Vaisman manifolds, Sasakian manifolds and so on. For nearly Kähler manifolds, the Hodge relations were sufficient to obtain the Hodge decomposition. During the work on Hodge theory of G_2-manifolds, many concepts of complex algebraic geometry were adapted to work on G_2-manifolds. This way I obtained some basic results in the theory of calibrated plurisubharmonic functions, later rediscovered by Harvey and Lawson in a different (and more systematic) framework.

This theory was used to study coherent sheaves on hyperkähler manifolds, and (jointly with Semyon Alesker) Calabi-Yau problem in HKT geometry, which became a much-researched subject since then.

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1.6 Locally conformally Kähler geometry

A **locally conformally Kähler manifold** (LCK-manifold) is a complex manifold which is covered by a Kähler, with the deck transform acting by holomorphic homotheties. An important special case is so-called **Vaisman manifolds**, which are covered by a Kähler manifold where $\mathbb{R}$ acts by holomorphic homotheties. Similarly one defines a locally conformally hyperkähler manifold. In 2004, I obtained a structure theorem for locally conformally hyperkähler manifolds, reducing their classification to classification of 3-Sasakian manifolds, which is due to Boyer, Galicki, Demailly and Kollar.

Since then, I collaborated with Liviu Ornea, and jointly we developed the subject to a great extent.

We have established a structure theorem for Vaisman manifolds, reducing the Vaisman geometry to Sasakian geometry, and proved that a Vaisman manifold admits a holomorphic immersion in a linear Hopf manifold. This was used to obtain similar immersion results for Sasakian manifold, proving that they always admit a CR-holomorphic embedding to a contact sphere. These results were also used to characterize CR-manifolds admitting Sasakian structure in terms of their automorphism group.

In attempt to understand the immersion theorem, we invented a new class of LCK-manifold, called **LCK-manifolds with automorphic potential**. This is an intermediate class between the Vaisman manifolds and LCK-manifolds. Unlike the Vaisman and LCK-manifolds, LCK-manifolds with automorphic potential are stable under small complex deformations. Also, such manifolds admit holomorphic embedding to linear Hopf manifold. This result can be understood as a locally conformally Kähler version of a Kodaira embedding theorem.

Later on, we found that LCK-manifolds with automorphic potential can be characterized in terms of vanishing of a certain cohomology class, which can be understood as a holomorphic version of Morse-Novikov cohomology. This was used to characterize such manifolds in terms of their automorphism groups, and to obtain important results about topology, describing their topology in terms of topology of certain algebraic varieties.

Our research culminated in a book, "Principles of Locally Conformally Kähler Geometry", by L. Ornea and M. Verbitsky, 769 pages, arXiv:2208.07188, 2022, which lays the foundation of modern LCK geometry in a way which should make the subject accessible to graduate students and research mathematicians.

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