Research overview

Misha Verbitsky, June 2009[1]

1 Hodge theory on hyperkähler manifolds and its applications

In [V90], [V94], [V95:1], [V95:2], I studied the applications of Hodge theory for topology of hyperkähler manifolds. It was shown that cohomology of a hyperkähler manifold admit an action of the Lie group $Sp(1,1)$, which is similar to Lefschetz' $SL(2)$-action. This was used to compute the cohomology algebra of a hyperkähler manifold, showing that its part generated by $H^2(M)$ is symmetric, up to the middle dimension. Among applications of these results, a proof of Mirror Conjecture for hyperkähler manifolds, and better understanding of hyperkähler subvarieties of hyperkähler manifolds and coherent sheaves.

2 Trianalytic subvarieties of hyperkähler manifolds

The papers [V93], [V94], [V96:1], [V96:2], [V97:1], [V97:2], [KV98:1], [V98], [KV98:2] and [V03:4] deal with trianalytic subvarieties in hyperkähler manifolds. These are subvarieties which are complex analytic with respect to three complex structures I, J, K. It was shown that all complex subvarieties of a generic hyperkähler manifold are trianalytic. Also all deformations of trianalytic subvarieties are again trianalytic, and their deformation space is singular hyperkähler.

I have studied singularities of singular hyperkähler varieties, and shown that a normalization of such variety is smooth and hyperkähler. This applies to trianalytic subvarieties, which are examples of singular hyperkähler spaces.

These results were applied to Hilbert schemes of points on K3 and generalized Kummer varieties. For Hilbert schemes of points on K3, it was shown that its generic deformation has no subvarieties. For generalized Kummer varieties, a similar attempt (joint with D. Kaledin) was foiled by our imperfect understanding of birational geometry of holomorphic symplectic manifolds. Soon after publishing this paper, we found a counterexample to one of our statements.

The birational geometry of holomorphic symplectic manifolds was studied by D. Kaledin in several papers in much detail. Our joint work in this direction resulted in [KV00], where a deformation theorem, analoguous to

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Bogomolov-Tian-Todorov, was obtained for non-compact holomorphic symplectic manifolds. Also, in [V99] I have studied quotient singularities of holomorphic symplectic manifolds admitting a holomorphic symplectic resolution, and proved that such singularities are always quotients by groups generated by symplectic reflections.

3 Coherent sheaves on hyperkähler manifolds

The papers [V92], [KV96], [V97:3], [V01:1], [MV06], are about hyperholomorphic bundles on hyperkähler manifolds. These are bundles with Hermitian connection with curvature of type (1,1) with respect to all complex structures induced by the hyperkähler structure.

It was shown that all stable bundles on generic hyperkähler manifolds admit a hyperholomorphic connection, which is unique. Conversely, every hyperholomorphic bundle is a direct sum of stable bundles.

The moduli spaces of such bundles wre shown to be singular hyperkähler, and the deformation unobstructed (except the first obstruction, known as Yoneda product).

Later, this notion was extended to coherent sheaves. Using results about trianalytic subvarieties, I have shown that a deformation of a hyperholomorphic bundle over a generic hyperkähler manifold M remains non-singular, unless M contains trianalytic subvarieties of complex codimension 2 (in the cases of a Hilbert scheme of K3 and the generalized Kummer variety, all trianalytic subvarieties have codimension > 2, except for 4-dimensional generalized Kummer).

If the deformation spaces of hyperholomorphic bundles are positive-dimensional (which is unknown yet), this should lead to new examples of hyperkähler manifolds.

In [KV96], a non-Hermitian version of this notion was studied. We have shown that if the Hermitian assumption is dropped, the non-Hermitian hyperholomorphic connection on M becomes essentially the same as a holomorphic structure on the lifting of the corresponding bundle to the twistor space. In [V02], [V03:3] this approach was used to study the category of coherent sheaves on generic K3 surfaces and tori. It was shown that this category is independent from the choice of a generic K3 or a torus of a given dimension.

4 Hodge theory on hypercomplex manifolds and HKT-geometry

A **hypercomplex manifold** is a manifold with three complex structures I, J, K satisfying quaternionic relations. It is called **quaternionic Her-**

mitian if it has a quaternionic-invariant Riemannian structure. With each quaternionic Hermitian manifold (M, I, J, K, g), one can associate its canonical $(2, 0)$-form $\Omega = \omega_J + \sqrt{-1}\omega_K$. If this form is closed, M is called hyperkähler (this is one of possible definitions). If $\partial\Omega = 0$, (M, I, J, K, g) is called **an HKT-manifold** ("hyperkähler with torsion").

An HKT form is in many ways similar to a Kähler structure. One can define a potential, a version of Hodge theory, Kähler class and so on. In papers [V01:2], [V03:2], [V04:2], [V04:3], [V06:1], [BDV07] I studied hypercomplex geometry from this point of view. The Hodge theory (including Lefschetz-type $SL(2)$-action) was constructed for HKT manifolds with trivial canonical bundle. As an application, it was shown that a compact hypercomplex manifold which admits a Kähler metric also admits a hyperkähler structure. In another application (jointly with I. Dotti and M. L. Barbieris) it was shown that a hypercomplex nilmanifold admits an HKT structure if and only if it is abelian.

This result was also useful in hyperkähler geometry, where a strong vanishing result was shown, based on this Lefschetz -type $SL(2)$-action. It was shown that cohomology of a holomorphic line bundle L with $-c_1(L)$ outside of a dual Kähler cone vanish after the middle dimension. Moreover, form any holomorphic bunldle M, $H^i(B \otimes L^N) = 0$, for N sufficiently big, and $i > \frac{1}{2} \dim_{\mathbb{C}} M$.

The version of Hodge theory developed for the study of HKT manifolds, was also useful in other geometric sutuations, namely, for G_2-manifolds and nearly Kähler manifolds ([V05:1], [V05:2], [V05:3]). For nearly Kähler manifolds, the Hodge relations were sufficient to obtain the Hodge decomposition. During the work on Hodge theory of G_2-manifolds, many concepts of complex algebraic geometry were adapted to work on G_2-manifolds. This way I obtained some basic results in the theory of calibrated plurisubharmonic functions, later rediscovered by Harvey and Lawson in a different (and more systematic) framework.

This theory was used to study coherent sheaves on hyperkähler manifolds, and (jointly with Semyon Alesker) Calabi-Yau problem in HKT geometry.

5 Plurisubharmonic functions in hypercomplex geometry

In [AV05], [AV08] we studied the plurisubharmonic function on a hypercomplex manifold M. If M is $\mathbb{H}^n$, these are functions which are subharmonic on all 1-dimensional quaternionic planes. The theory of quaternionic plurisubharmonic functions is deeply related with HKT-geometry, because HKT-potentials are precisely the C^2-functions which are strictly plurisubharmonic. We formulated a version of Calabi conjecture for HKT manifolds and proved uniqueness of its solution and C^0-estimates. In [V08:3], it was shown that an HKT metric is Calabi-Yau HKT if and only if it is balanced (satisfies $d\omega^{\dim_{\mathbb{C}} M-1} = 0$).

The appropriate notion of positivity (called K-positivity there) originates in [V01:1], where it was used to study direct image of hyperholomorphic sheaves. In order to prove stability, an L^2-estimation of singularities was required. It was obtained by a clumsy approximation argument. In [V08:1] and [V08:2] the theory of calibrated plurisubharmonic function was developed, in parallel with the usual complex analysis, and the stability of higher direct images of hyperholomorphic sheaves was obtained in a clean way as an application of this theory.

6 Locally conformally Kähler geometry

A **locally conformally Kähler manifold** (LCK-manifold) is a complex manifold which is covered by a Kähler, with the deck transform acting by holomorphic homotheties. An impotrant special case is so-called **Vaisman manifolds**, which are covered by a Kähler manifold where $\mathbb{R}$ acts by holomorphic homotheties. Similarly one defines a locally conformally hyperkähler manifold. In [V03:1], I obtained a structure theorem for locally conformally hyperkähler manifolds, reducing their classification to classification of 3-Sasakian manifolds, which is due to Boyer, Galicki, Demailly and Kollar.

Since then, I collaborated with Liviu Ornea in a series of papers ([OV03:1], [OV03:12], [OV04], [OV06:1], [OV06:2], [OV06:3], [OV07:1], [OV09:1], [OV09:2]) on locally conformally Kähler geometry.

We have established a structure theorem for Vaisman manifolds, reducing the Vaisman geometry to Sasakian geometry, and proved that a Vaisman manifold admits a holomorphic immersion in a linear Hopf manifold. This was used to obtain similar immersion results for Sasakian manifold, proving that they always admit a CR-holomorphic embedding to a contact sphere. These results were also used to characterize CR-manifolds admitting Sasakian structure in terms of their automorphism group.

In attempt to understand the immersion theorem, we invented a new class of LCK-manifold, called **LCK-manifolds with automorphic potential**. This is an intermediate class between the Vaisman manifolds and LCK-manifolds. Unlike the Vaisman and LCK-manifolds, LCK-manifolds with automorphic potential are stable under small complex deformations. Also, such manifolds admit holomorphic embedding to linear Hopf manifold. This result can be understood as a locally conformally Kähler version of a Kodaira embedding theorem.

Later on, we found that LCK-manifolds with automorphic potential can be characterized in terms of vanishing of a certain cohomology class, which can be understood as a holomorphic version of Morse-Novikov cohomology. This was used to characterize such manifolds in terms of their automorphism groups, and to obtain important results about topology, describing their topology in terms of topology of certain algebraic varieties.

7 Toric and elliptic fibrations

In [V04:1], I studied vector bundles and coherent sheaves on compact complex non-Kähler manifolds admitting a principal elliptic fibration with a Kähler base. There are many examples of such manifolds coming from physics, hypercomplex geometry and LCK-geometry. Positive elliptic fibrations were defined (a big class including quasi-regular Vaisman and Calabi-Eckmann manifolds). I have shown that any stable sheaf on a positive elliptic fibration of $\dim_{\mathbb{C}} \geq 3$ is lifted from a base (up to a product with line bundle). This implies, in particular, that all coherent sheaves are filtrable (in constract to the case of non-Kähler surfaces, where coherent sheaves are rarely filtrable). In [V04:4], filtrability was generalized to Hopf manifolds of $\dim_{\mathbb{C}} \geq 3$. In [V07:1], the notion of positive toric fibration was discussed, including the positive elliptic fibrations and invariant complex structures on compact Lie groups. It was shown that all connected subvarieties in a positive toric fibration are contained in a fiber, or lifted from a base.

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