STABLE SHEAVES AND EINSTEIN-HERMITIAN METRICS

SHIGETOSHI BANDO
Mathematical Institute, Tohoku University, Sendai, 980-77, Japan

and

YUM-TONG SIU
Department of Mathematics, Harvard University, Cambridge, MA02138, USA

After the pioneering work of Kobayashi (cf. [8]), the relation between stable holomorphic vector bundles and Einstein-Hermitian connections is now quite well-understood by works of Donaldson, Narasimhan, Seshadri, Simpson, Uhlenbeck, Yau and others (cf. [3, 4, 5, 9, 12, 13, 15, 16, 17, 18, 19, 20]). The purpose of this paper is to generalize the results for vector bundles to the case of reflexive sheaves by extending the notion of "Einstein-Hermitian metrics".

We recall the definition of stable sheaves and introduce the notion of admissible Hermitian metrics on torsion free sheaves.

Definition. Let $\mathcal{E}$ be a torsion free coherent analytic sheaf on an n-dimensional compact Kähler manifold (X, ω) with the fundamental form ω. We define the slope $\mu(\mathcal{E})$ of $\mathcal{E}$ by

$$\mu(\mathcal{E}) = (c_1(\mathcal{E}) \cup [\omega]^{n-1})[X]/\text{rank}\mathcal{E}.$$

1) $\mathcal{E}$ is called (semi-)stable, if for any subsheaf $\mathcal{S}$ such that $0 < \text{rank}\mathcal{S} < \text{rank}\mathcal{E}$ we have $\mu(\mathcal{S}) < \mu(\mathcal{E})$ ($\mu(\mathcal{S}) \leq \mu(\mathcal{E})$), respectively.
2) $\mathcal{E}$ is called poly-stable, if it is a direct sum of stable sheaves $\mathcal{E}_i$ with the same slope $\mu(\mathcal{E}_i) = \mu(\mathcal{E})$.
3) A Hermitian metric h of $\mathcal{E}$ defined on the locally free part of $\mathcal{E}$ is called admissible, if its curvature tensor F is square integrable and its trace ΛF with respect to the base metric ω is uniformly bounded.

The admissible Hermitian metrics play the role of the ordinary Hermitian metrics for vector bundles.

Theorem 1. *Any reflexive sheaf $\mathcal{E}$ on an n-dimensional compact Kähler manifold (X, ω) admits an admissible Hermitian metric.*

The meaning of admissibleness of a Hermitian metric is understood by the following

Theorem 2. *Let (E, h) be a holomorphic vector bundle with a Hermitian metric h defined on a Kähler manifold (Y, ω) (not necessary compact nor complete) outside a closed subset S with locally finite Hausdorff measure of real co-dimension 4. Assume that its curvature tensor F is locally square integrable on Y, then*

a) *E extends to the whole space Y as a reflexive sheaf $\mathcal{E}$, and for any local section $s \in \Gamma(U, \mathcal{E})$, $\log^+ h(s, s)$ belongs to H^1_{loc}.*

b) *If ΛF is locally bounded, then $h(s,s)$ is locally bounded, and h belongs to $L^p_{2\,\mathrm{loc}}$ for any finite p where $\mathcal{E}$ is locally free.*
c) *If (E,h) is Einstein-Hermitian, then h smoothly extends as an Einstein-Hermitian metric over the place where $\mathcal{E}$ is locally free.*

This is a combination of the removable singularity theorem [1] and the slicing theorem [14]. Theorem 2 and the compactness theorem for Yang-Mills connections [10] yield the following Corollary which generalizes the results of Uhlenbeck [16], [17] to the higher dimensional case.

Corollary 2. *Let (E_i,h_i) be a sequence of Einstein-Hermitian holomorphic vector bundles with fixed first and second Chern classes on a compact Kähler manifold (X,ω). Then there exist a subsequence, which we still call (E_i,h_i), and a reflexive sheaf $\mathcal{E}$ with an admissible Einstein-Hermitian metric h such that under suitable gauge change (E_i,h_i) smoothly converges to $(\mathcal{E},h)$ outside a closed subset with finite Hausdorff measure of real co-dimension 4.*

For examples of convergence to sheaves in higher dimension, see [11].
Our main result is

Theorem 3. *A reflexive sheaf $\mathcal{E}$ on an n-dimensional compact Kähler manifold (X,ω) admits an admissible Einstein-Hermitian metric, if and only if $\mathcal{E}$ is poly-stable.*

Corollary 3. *For a poly-stable reflexive sheaf $\mathcal{E}$, we have the Bogomolov inequality*

$$(2rc_2(\mathcal{E})-(r-1)c_1(\mathcal{E})^2)\cup[\omega]^{n-2}[X]\geq 0,$$

where $r=\mathrm{rank}\mathcal{E}$. And the equality holds if and only if $\mathcal{E}$ is locally free and its Einstein-Hermitian metric gives a projectively flat connection.

For non-stable ones we have the following result.

Theorem 4. *Let $\mathcal{E}$ be a reflexive sheaf on (X,ω). If $\mathcal{E}$ is not stable, then it "breaks up" into a direct sum of Einstein-Hermitian sheaves via the heat equation considered below.*

It is an interesting question whether the sum is isomorphic to the reflexization of $\mathrm{Gr}\mathcal{E}$ or not.

Remark. We can work with torsion free sheaves. But since it would result in some complication of the statement and since metrics essentially deal with the reflexization, we restrict ourselves to reflexive sheaves.

The idea of the proof of Theorem 1, 3 and 4 is as follows. First let us assume that X is projective algebraic. Then there exits a resolution $\cdots\longrightarrow E_1^\vee\longrightarrow E_0^\vee\longrightarrow\mathcal{E}^\vee\longrightarrow 0$ of the dual sheaf $\mathcal{E}^\vee$ of a reflexive sheaf $\mathcal{E}$ on (X,ω) by holomorphic vector bundles. Taking its dual $0\longrightarrow\mathcal{E}\longrightarrow E_0\longrightarrow E_1$, we get an embedding of $\mathcal{E}$ into a holomorphic vector bundle E_0 as a holomorphic subbundle outside S where $\mathcal{E}$ fails to be locally free. We fix an arbitrary Hermitian metric h_0 on E_0. It induces a Hermitian metric h on $\mathcal{E}$ outside S. For the general case, we perform the above construction locally, and patch the locally defined Hermitian metrics to get the global h. We deform h by the heat equation

$$\frac{dh}{dt}h^{-1} = -(\sqrt{-1}\Lambda F - \lambda(\mathcal{E})I),$$

where $\lambda(\mathcal{E}) = 2\pi n\mu(\mathcal{E})/[\omega]^n[X]$ and I stands the identity endomorphism of the fibre. We show that the solution for $t > 0$ is an admissible one. If $\mathcal{E}$ is stable, we obtain an admissible Einstein-Hermitian metric as the limit as t tends to infinity.

1. Removable singularity theorem

Here we consider Theorem 2. As the theorem is of local nature, we can assume Y to be a domain in $\mathbf{C}^n$, S of finite Hausdorff measure of real co-dimension 4 and F square integrable.

The following is a direct consequence of the removable singularity theorem [1] and the slicing theorem [14].

Lemma 1. *Let (E, h) be a Hermitian holomorphic vector bundle defined on the complement of a closed subset S of a product of balls $\mathbf{B}^2 \times \mathbf{B}^{n-2} \subset \mathbf{C}^2 \times \mathbf{C}^{n-2}$ Assume that S has finite Hausdorff measure of real co-dimension 4 and there exists a compact subset K of $\mathbf{B}^2$ such that S is contained in $K \times \mathbf{B}^{n-2}$ If the curvature tensor of (E, h) is square integrable, then E extends to the whole space $\mathbf{B}^2 \times \mathbf{B}^{n-2}$ as a reflexive sheaf.*

Let us first consider a) of Theorem 2. Fix an arbitrary point, say 0, in Y. Then by assumption, for a generic projection $p : Y \subset \mathbf{C}^n \longrightarrow \mathbf{C}^{n-2}$ the set $S \cap p^{-1}(0)$ consists of a countable number of points which may accumulate only at 0. Shrinking the domain Y, we may assume we are in the situation of Lemma 1. Thus E extends as a reflexive sheaf $\mathcal{E}$. Fix an arbitrary section $s \in \Gamma(Y, \mathcal{E})$. We put $Y_t = p^{-1}(t)$, $S_t = S \cap Y_t$ and $u_t = \log^+ h(s, s)|_{Y_t}$. Let ϕ be a smooth function on $\mathbf{B}^2$ with compact support which equals 1 on a neighborhood of K. Except for t which belongs to a set of measure zero, Y_t contains only a finite number of points in S_t and the restriction F_t of F to Y_t is square integrable. We fix such a point t. In section 3 of [1] we essentially showed that u_t belongs to H^1_{loc} and satisfies the following inequality

$$\triangle_t u_t \geq -4|F_t|,$$

where $\triangle_t$ is the Laplacian on Y_t. Then we get for any $\epsilon > 0$

$$\begin{aligned}\int_{Y_t} |\nabla(\phi u_t)|^2 &\leq \int_{Y_t} 4|F_t|\phi^2 u_t + \int_{Y_t} |\nabla\phi|^2 u_t^2 \\ &\leq \epsilon \int_{Y_t} (\phi u_t)^2 + 4\epsilon^{-1} \int_{Y_t} \phi^2 |F_t|^2 + \int_{Y_t} |\nabla\phi|^2 u_t^2.\end{aligned}$$

We apply the Poincaré inequality and get

$$\int_{Y_t} |\nabla(\phi u_t)|^2 \le 2 \int_{Y_t} |\nabla\phi|^2 u_t^2 + C \int_{Y_t} \phi^2 |F_t|^2$$

Here and hereafter we denote a general positive constant by C which may differ in the different appearance. We integrate it in t and get for a compact subset $K' \subset \mathbf{B}^{n-2}$

$$\int_{p^{-1}(K')} |\nabla'(\phi u_t)|^2 \le 2 \int_{p^{-1}(K')} |\nabla\phi|^2 u_t^2 + C \int_{p^{-1}(K')} \phi^2 |F_t|^2,$$

where ∇' stands the derivation along the fibres of the projection p. Since we can use any generic projection p, it shows the desired result $u \in H^1_{\mathrm{loc}}$.

Once $\log^+ h(s,s) \in H^1_{\mathrm{loc}}$ is known, it is easy to see that $\Lambda F \in L^\infty_{\mathrm{loc}}$ implies $\log^+ h(s,s) \in L^\infty_{\mathrm{loc}}$ in view of

$$\triangle \log^+ h(s,s) \ge -2|\Lambda F|.$$

c) is a consequence of b) and the standard regularity theorem of linear elliptic partial differential equations.

The last half of b) follows from

Proposition 1. *Let h be a rank r Hermitian matrix valued function defined on an n-dimensional Kähler manifold (Y, ω) which belongs to H^1 Assume that h and h^{-1} are uniformly bounded and it satisfies*

$$\Lambda\bar\partial(\partial h h^{-1}) = f \tag{1}$$

in a weak sense with a uniformly bounded function f, then h belongs to $C^{1,\alpha}_{\mathrm{loc}}$ for any $0 < \alpha < 1$ and admits an estimate depending only on $\|h\|_{L^\infty}$, $\|h^{-1}\|_{L^\infty}$, $\|f\|_{L^\infty}$ and the geometry of (Y, ω).

One can show Proposition 1 by modifying the argument which was used to show the corresponding result for harmonic mappings in [7]. Since the proof is essentially the same, we restrict ourselves only to pointing out the places where change is necessary. Again we work locally. We denote the complex Laplacian by $\square = \Lambda\sqrt{-1}\partial\bar\partial$ and its Green's function by $G(x, y)$.

Step 1: h is continuous. The equation (1) means that for any matrix valued function $k \in L^\infty \cap H^1$ with compact support we have

$$\int \mathrm{tr}\partial h h^{-1} \bar\partial k \omega^{n-1} + \int \mathrm{tr} f k \omega^n = 0.$$

Let ϕ be a cut off function such that $\phi = 1$ on $B(x_0, \rho)$, $= 0$ outside $B(x_0, 2\rho)$ and $|\nabla\phi| < 2\rho^{-1}$. We substitute $k = \phi G(\cdot, x_0)h$ and get the estimate

$$\int_{B(x_0,\rho)} G(\cdot, x_0)\partial h h^{-1}\bar{\partial} h\omega^{n-1} \le C\|h\|_{L^\infty}(1 + \|f\|_{L^\infty}\rho^2)$$

as in [7]. Thus for any $0 < \rho \ll 1$ there exists ρ_0 such that $\rho < \rho_0 < \sqrt{\rho}$ and

$$\frac{1}{\rho_0^{2n-2}} \int_{B(x_0,\rho_0)} |\nabla h|^2 \le \frac{C}{-\log \rho}.$$

Let h_0 be the average of h on the ball $B(x_0, \rho_0)$. Then there exists a positive constant C such that

$$\Box(\mathrm{tr}_h h_0 + \mathrm{tr}_{h_0} h - 2r) \ge -C.$$

Hence applying the Poincaré inequality we obtain

$$\begin{aligned}\sup_{B(x_0,2^{-1}\rho)} |h - h_0|^2 &\le C \sup_{B(x_0,2^{-1}\rho_0)} (\mathrm{tr}_h h_0 + \mathrm{tr}_{h_0} h - 2r) \\ &\le C(\rho_0^{-2n} \int_{B(x_0,\rho_0)} |h - h_0|^2 + \rho_0^2) \\ &\le C(\rho_0^{2-2n} \int_{B(x_0,\rho_0)} |\nabla h|^2 + \rho_0^2) \\ &\le C(\frac{1}{-\log \rho} + \rho).\end{aligned}$$

Step 2: h is Hölder continuous. Fix a sufficiently small positive number ρ and let h_0 be the average of h on $B(x_0, 2\rho)\backslash B(x_0, \rho)$. Since h is continuous, there exist positive constants a and C such that in a neighborhood of x_0 it holds

$$\Box|h - h_0|^2 \ge a|\nabla h|^2 - C.$$

Multiplying it by $\phi G(\cdot, x_0)$, integrating by parts and again applying the Poincaré inequality, we obtain

$$\begin{aligned}\rho^{2-2n} \int_{B(x_0,\rho)} |\nabla h|^2 &\le C \int_{B(x_0,\rho)} G(\cdot, x_0)|\nabla h|^2 \\ &\le C(\rho^{-2n} \int_{B(x_0,2\rho)\backslash B(x_0,\rho)} |h - h_0|^2 + \rho^2) \\ &\le C(\rho^{2-2n} \int_{B(x_0,2\rho)\backslash B(x_0,\rho)} |\nabla h|^2 + \rho^2),\end{aligned}$$

which implies the Hölder continuity.

Step 3: h belongs to $C^{1,\alpha}$. Instead of the formula (6.32) in the page 84 in [7], we apply the following formula. For $k \in H^1$ with compact support

$$\int \mathrm{tr}\partial h\bar{\partial} k\omega^{n-1} + \int \mathrm{tr}\partial h(h^{-1}h_0 - I)\bar{\partial} k\omega^{n-1} + \int \mathrm{tr} f h_0 k\omega^n = 0,$$

with $h_0 = h(x_0)$. Then the argument in [7] shows the desired result.

2. Heat kernel estimate

For later use, we show the uniform boundedness of heat kernels for a certain type of degeneration of metrics.

Proposition 2. *Let (X,ω) be an n-dimensional compact Kähler manifold and $\pi : Y \longrightarrow X$ a blowing up with non-singular center. Fix an arbitrary Kähler metric θ on Y and set $\omega_\epsilon = \pi^*\omega + \epsilon\theta$ for $0 < \epsilon \leq 1$. Let H_ϵ be the heat kernel with respect to the metric ω_ϵ, then we have a uniform estimate $0 \leq H_\epsilon \leq C(t^{-n}+1)$ with a positive constant C.*

For the proof we start with the following lemma.

Lemma 2. *If a real n-dimensional Riemannian manifold (M,g) satisfies the Sobolev inequality*

$$\Big(\int |\phi|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \leq S\int |\nabla\phi| \qquad \text{for } \ \phi \in C_0^1(M),$$

then the product manifold $M \times \mathbf{R}$ also satisfies the Sobolev inequality with the Sobolev constant $S^{n/n+1}$

Proof. For an arbitrary point (x,t) in $M\times\mathbf{R}$ and a function $\phi \in C_0^1(M\times\mathbf{R})$, we have

$$|\phi(x,t)| \leq \int_{\mathbf{R}} |\nabla\phi(x,s)|\,ds$$

and

$$\Big(\int_M |\phi(y,t)|^{\frac{n}{n-1}}dy\Big)^{\frac{n-1}{n}} \leq S\int_M |\nabla\phi(y,t)|\,dy.$$

Thus

$$\begin{aligned}\int_{M\times\mathbf{R}} |\phi|^{\frac{n+1}{n}} &= \int_{\mathbf{R}} dt \int_M |\phi(y,t)|^{\frac{n+1}{n}}dy \\ &\leq \int_{\mathbf{R}} dt\Big(\int_M |\phi(y,t)|^{\frac{n}{n-1}}dy\Big)^{\frac{n-1}{n}}\Big(\int_M |\phi(y,t)|\,dy\Big)^{\frac{1}{n}} \\ &\leq S\int_{\mathbf{R}} dt\int_M |\nabla\phi|(y,t)\,dy\Big(\int_M dy\int_{\mathbf{R}} |\nabla\phi|(y,s)\,ds\Big)^{\frac{1}{n}} \\ &= S\Big(\int_{M\times\mathbf{R}} |\nabla\phi|\Big)^{\frac{n+1}{n}}\end{aligned}$$

Lemma 3. *Let (X,ω) be an n-dimensional compact Kähler manifold and $\pi : Y \longrightarrow X$ a blowing up with non-singular center. Fix an arbitrary Kähler metric θ on Y and set $\omega_\epsilon = \pi^*\omega + \epsilon\theta$ for $0 < \epsilon \leq 1$. Then (Y,ω_ϵ) satisfies the Sobolev inequality*

$$\Big(\int |\phi|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \leq S\int |\nabla\phi| + |\phi| \qquad \text{for } \ \phi \in C^1(Y),$$

with S independent of ϵ.

Proof. We denote the blowing up of $\mathbf{C}^k$ at the origin by $\hat{\mathbf{C}}^k$ and its restriction to a ball $\mathbf{B}^k$ by $\hat{\mathbf{B}}^k$. We fix an arbitrary point p in X. Then in a neighborhood of p the blowing up is given by $\pi : \hat{\mathbf{B}}^k \times \mathbf{B}^l \longrightarrow \mathbf{B}^k \times \mathbf{B}^l$ for some k, l. We first show the uniform Sobolev inequality for the functions ϕ whose supports are contained in $\hat{\mathbf{B}}^k \times \mathbf{B}^l$:

$$\left(\int |\phi|^{\frac{2n}{2n-1}}\right)^{\frac{2n-1}{2n}} \leq S \int |\nabla\phi|.$$

For two metrics g_0, g_1 such that $c_1 g_0 \leq g_1 \leq c_2 g_0$, the corresponding Sobolev constants S_0, S_1 satisfy $(c_1/c_2)^n S_0 \leq S_1 \leq (c_2/c_1)^n S_0$. Hence it is sufficient to consider the family of metrics $\sqrt{-1}\partial\bar{\partial}(|z|^2 + \epsilon \log|z|^2 + |w|^2)$, where (z,w) is the coordinate of $\mathbf{B}^k \times \mathbf{B}^l$. Changing the coordinate by $(z,w) \longrightarrow \sqrt{\epsilon}(z,w)$, we reduce the problem to the verification of Sobolev inequality on $(\hat{\mathbf{C}}^k \times \mathbf{C}^l, \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log|z|^2 + |w|^2))$. Since that for $(\hat{\mathbf{C}}^k, \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log|z|^2))$ is trivial, Lemma 2 implies the uniform Sobolev inequality on $\hat{\mathbf{B}}^k \times \mathbf{B}^l$. Let ρ_i be a partition of unity on X corresponding to the covering given by $\mathbf{B}^k \times \mathbf{B}^l$'s considered above. Then, *since it holds* $\sup|\nabla\pi^*\rho_i|_{\omega_\epsilon} \leq \sup|\nabla\rho_i|_\omega$, *summing up the Sobolev inequalities for* $(\pi^*\rho_i)\phi$ we get the desired uniform Sobolev inequality on Y

The following result obtained by Cheng and Li [2] concludes the proof of Proposition 2.

Lemma 4. *If a real n-dimensional compact Riemannian manifold (M,g) satisfies the Sobolev inequality*

$$\left(\int |\phi|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq S \int |\nabla\phi| + |\phi| \qquad \text{for } \phi \in C^1(M),$$

then the heat kernel H satisfies the estimate $0 \leq H \leq C(t^{-n/2}+1)$ with a positive constant C which depends only on n and S.

3. Existence of admissible Hermitian metrics

Let $\mathcal{E}$ be a rank r reflexive sheaf on an n-dimensional compact Kähler manifold (X,ω). We take a finite cover $\{U_\alpha\}$ and local resolutions $\cdots \longrightarrow E^\vee_{1,\alpha} \longrightarrow E^\vee_{0,\alpha} \longrightarrow \mathcal{E}^\vee|U_\alpha \longrightarrow 0$ of the dual sheaf $\mathcal{E}^\vee$ by holomorphic vector bundles, which admits the following commutative diagram.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\phi^\vee_{1,\alpha}} & E^\vee_{1,\alpha}|U_\alpha \cap U_\beta & \xrightarrow{\phi^\vee_{0,\alpha}} & E^\vee_{0,\alpha}|U_\alpha \cap U_\beta & \xrightarrow{\phi^\vee_\alpha} & \mathcal{E}^\vee|U_\alpha \cap U_\beta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi^\vee_{\alpha\beta}} & & \| & & \\
\cdots & \xrightarrow{\phi^\vee_{1,\beta}} & E^\vee_{1,\beta}|U_\alpha \cap U_\beta & \xrightarrow{\phi^\vee_{0,\beta}} & E^\vee_{0,\beta}|U_\alpha \cap U_\beta & \xrightarrow{\phi^\vee_\beta} & \mathcal{E}^\vee|U_\alpha \cap U_\beta & \longrightarrow & 0
\end{array}$$

Taking its dual we get the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{E}|U_\alpha \cap U_\beta & \overset{\phi_\alpha}{\longrightarrow} & E_{0,\alpha}|U_\alpha \cap U_\beta & \overset{\phi_{0,\alpha}}{\longrightarrow} & E_{1,\alpha}|U_\alpha \cap U_\beta \\
 & & \| & & \downarrow{\scriptstyle \phi_{\beta\alpha}} & & \downarrow \\
0 & \longrightarrow & \mathcal{E}|U_\alpha \cap U_\beta & \overset{\phi_\beta}{\longrightarrow} & E_{0,\beta}|U_\alpha \cap U_\beta & \overset{\phi_{0,\beta}}{\longrightarrow} & E_{1,\beta}|U_\alpha \cap U_\beta
\end{array}$$

This allows us to see $\mathcal{E}$ locally as a subsheaf of locally defined holomorphic vector bundles $E_{0,\alpha}$. The relfexive sheaf $\mathcal{E}$ is locally free outside a subvariety S of co-dimension at least 3 and outside S the embedding $\mathcal{E} \longrightarrow E_{0,\alpha}$ realizes $\mathcal{E}$ as a holomorphic subbundle.

We take arbitrary Hermitian metrics $\tilde{h}_\alpha$ on $E_{0,\alpha}$ and a partition of unity ρ_α subject to the covering $\{U_\alpha\}$. Set $h_\alpha = \sum \rho_\beta \phi^*_{\beta\alpha} \tilde{h}_\beta$. Then, the Hermitian metrics $\{h_\alpha\}$ on $\{E_{0,\alpha}\}$ give the same pull back metric $\phi^*_\alpha h_\alpha$. and define a Hermitian metric h on $\mathcal{E}$ outside S. We deform h to get an admissible Hermitian metric on $\mathcal{E}$ by the heat equation taking h as the initial metric.

$$\frac{dh}{dt} h^{-1} = -(\sqrt{-1}\Lambda F - \lambda(\mathcal{E})I), \tag{2}$$

where $\lambda(\mathcal{E}) = 2\pi n \mu(\mathcal{E})/[\omega]^n[X]$ and I stands the identity endomorphism of the fibre. (To be precise, the heat equation is to be satisfied outside S.)

Since the equation has singularity on S, we make a regularization to solve it. We take blowing up with non-singular center finite times $\pi_i : X_i \longrightarrow X_{i-1}$, $(i = 1, \ldots k)$, $X_0 = X$ such that $\pi = \pi_k \pi_{k-1} \ldots \pi_1 : X_k \longrightarrow X$ is biholomorphic outside S and the kernel of $\pi^* \phi_{0,\alpha} : \pi^* E_{0,\alpha} \longrightarrow \pi^* E_{1,\alpha}$ gives a holomorphic subbundle E_α. E_α's are naturally identified on the intersection of U_α's, and we get a holomorphic vector bundle E on X_k. As in the case of $\mathcal{E}$, E naturally inherits a Hermitian metric from h_α's, which we still call h. Fix arbitrary Kähler metrics θ_i on X_i and define a family of Kähler metrics on X_i by $\omega_{i,\epsilon} = \omega_{i,\epsilon_1,\epsilon_2,\ldots,\epsilon_i} = \omega + \epsilon_1\theta_1 + \epsilon_2\theta_2 + \ldots + \epsilon_i\theta_i$ for positive numbers $\{\epsilon_j\}_{j=1}^k$. Here and hereafter we denote geometric objects and their pulling back by the same notation.

We consider the heat equation for Hermitian metrics of the holomorphic vector bundle E on $(X_k, \omega_{k,\epsilon})$ taking the above constructed h as the initial metric.

$$\frac{dh}{dt} h^{-1} = -(\sqrt{-1}\Lambda_{k,\epsilon} F - \lambda(\mathcal{E})I). \tag{2$_k$}$$

Here $\Lambda_{k,\epsilon}$ means the trace taken with respect to the metric $\omega_{k,\epsilon}$. Since every thing is now in the smooth category, the result of Donaldson [4] says that $(2)_k$ has a solution untill infinite time. The curvature tensor satisfies the following.

$$\frac{d\Lambda_{k,\epsilon} F}{dt} = \Box_{k,\epsilon} \Lambda_{k,\epsilon} F, \tag{3.1$_k$}$$

$$\frac{d|\Lambda_{k,\epsilon}F|^2}{dt} = \Box_{k,\epsilon}|\Lambda_{k,\epsilon}F|^2 - |\nabla\Lambda_{k,\epsilon}F|^2, \tag{3.2$_k$}$$

$$\frac{d|\Lambda_{k,\epsilon}F|}{dt} \leq \Box_{k,\epsilon}|\Lambda_{k,\epsilon}F|, \tag{3.3$_k$}$$

$$\frac{d}{dt}\int_{X_k}|\Lambda_{k,\epsilon}F|^2 = -\int_{X_k}|\nabla\Lambda_{k,\epsilon}F|^2, \tag{3.4$_k$}$$

$$\int_{y\in X_k}|\Lambda_{k,\epsilon}F|(t,y) \leq \int_{y\in X_k}|\Lambda_{k,\epsilon}F|(0,y), \tag{3.5$_k$}$$

$$|\Lambda_{k,\epsilon}F|(t,x) \leq \int_{y\in X_k} H_{k,\epsilon}(t,x,y)|\Lambda_{k,\epsilon}F|(0,y), \tag{3.6$_k$}$$

where $\Box_{k,\epsilon}$ and $H_{k,\epsilon}(t,x,y)$ are the complex (crude) Laplacian with respect to $\omega_{k,\epsilon}$ and its heat kernel.

Lemma 5. *Let F be the curvature tensor of the initial metric h which is induced by the metric h_α on $E_{0,\alpha}$. Then its trace $\Lambda_{i,\epsilon}F$ on $(X_i, \omega_{i,\epsilon})$, $0 < \epsilon_j \leq 1$, is uniformly integrable.*

Proof. Let $F_{0,\alpha}$ be the curvature tensor of the holomorphic Hermitian vector bundle $(E_{0,\alpha}, h_\alpha)$ and p_α the orthogonal projection onto $\mathcal{E}$. We take a subcovering $\{ V_\alpha \}$ of $\{ U_\alpha \}$ consisting of compact subdomains $V_\alpha \subset U_\alpha$. Then on each domain V_α we have the inequality $\sqrt{-1}F = \sqrt{-1}p_\alpha F_{0,\alpha}p_\alpha + \sqrt{-1}\bar\partial p_\alpha \wedge \partial p_\alpha \leq \sqrt{-1}p_\alpha F_{0,\alpha}p_\alpha \leq C\omega$ with some positive constant C, which shows that $\sqrt{-1}F$ is bounded from above on X_i. It holds

$$\begin{aligned}
|\Lambda_{i,\epsilon}F|\,\omega_{i,\epsilon}^n &\leq (|\Lambda_{i,\epsilon}(\sqrt{-1}F - C\omega I)| + |\Lambda_{i,\epsilon}C\omega I|)\,\omega_{i,\epsilon}^n \\
&\leq n\mathrm{tr}(2C\omega I - \sqrt{-1}F)\,\omega_{i,\epsilon}^{n-1} \\
&\leq n\mathrm{tr}(2C\omega I - \sqrt{-1}F)\,\omega_{k,\epsilon_1}^{n-1},
\end{aligned}$$

with $\epsilon_1 = (1,1,\ldots,1)$. And since F gives the curvature of the holomorphic vector bundle E on X_k,

$$\int_{X_k}\mathrm{tr}(2C\omega I - \sqrt{-1}F)\,\omega_{k,\epsilon_1}^{n-1} = (2Cr[\omega] - 2\pi c_1(E)) \cup [\omega_{k,\epsilon_1}]^{n-1}[X_k].$$

A direct calculation shows (cf. [8], [15])

Lemma 6. *Let (E,h) be a holomorphic Hermitian vector bundle of rank r on an n-dimensional compact Kähler manifold (X,ω). Then we have*

$$(2c_2(E) - c_1(E)^2) \cup [\omega]^{n-2}[X] = c_n\int_X(|F|^2 - |\Lambda F|^2)\,\omega^n,$$

$$(2c_2(E) - \frac{r-1}{r}c_1(E)^2) \cup [\omega]^{n-2}[X] = c_n\int_X(|F^0|^2 - |\Lambda F^0|^2)\,\omega^n,$$

where $c_n = (4\pi^2 n(n-1))^{-1}$ *and* F^0 *stands the trace free part.*

Proposition 2 says that for a fixed $\epsilon' = (\epsilon_1, \epsilon_2, \dots, \epsilon_{k-1})$ the heat kernel $H_{k,\epsilon}$ has a uniform bound for $0 < \epsilon_k \le 1$. It is easy to see that $H_{k,\epsilon}$ converges to $H_{k-1,\epsilon'}$ outside the exceptional set as ϵ_k tends to 0. By the inequalities $(3.5)_k$ and $(3.6)_k$, Lemma 5 implies that $|\Lambda_{k,\epsilon}F|$ has a uniform L^1 bound for $t \ge 0$ and a uniform L^∞ bound for $t \ge t_0 > 0$ or on a compact set disjoint from the exceptional set. Proposition 1 gives $C^{1,\alpha}$ estimate on h. Lemma 6 gives the uniform L^2 bound on F. Thus we can substruct a subsequence which converges to a solution of the heat equation on X_{k-1} as ϵ_k tends to 0,

$$\frac{dh}{dt}h^{-1} = -(\sqrt{-1}\Lambda_{k-1,\epsilon'}F - \lambda(\mathcal{E})I), \qquad (2)_{k-1}$$

such that F has a uniform L^2 bound for $t \ge 0$ and $\Lambda_{k-1,\epsilon'}F$ has a uniform L^1 bound for $t \ge 0$ and a uniform L^∞ bound for $t \ge t_0 > 0$. It holds

$$\frac{d}{dt}\int_{X_{k-1}} |\Lambda_{k-1,\epsilon'}F|^2 = -\int_{X_{k-1}} |\nabla\Lambda_{k-1,\epsilon'}F|^2, \qquad (3.4)_{k-1}$$

$$\int_{y\in X_{k-1}} |\Lambda_{k-1,\epsilon'}F|(t,y) \le \int_{y\in X_{k-1}} |\Lambda_{k-1,\epsilon'}F|(0,y), \qquad (3.5)_{k-1}$$

$$|\Lambda_{k-1,\epsilon'}F|(t,x) \le \int_{y\in X_{k-1}} H_{k-1,\epsilon'}(t,x,y)|\Lambda_{k-1,\epsilon'}F|(0,y). \qquad (3.6)_{k-1}$$

Again taking the limit $\epsilon_{k-1} \longrightarrow 0$, we have a solution of the heat equation on X_{k-2}. Repeating the argument, we obtain a solution of the heat equation (2) on X which gives the desired admissible Hermitian metric h for the positive time $t > 0$. Since $\mathcal{E}$ and E is isomorphic outside S which has at least co-dimension 3, by Fubini's theorem we get

$$(2c_2(\mathcal{E}) - \frac{r-1}{r}c_1(\mathcal{E})^2) \cup [\omega]^{n-2}[X] \ge c_n \int_X (|F^0|^2 - |\Lambda F^0|^2)\,\omega^n. \qquad (4)$$

And it holds

$$\int_{t_0}^{\infty}\int_X |\nabla\Lambda F|^2(t,y) \le \int_X |\Lambda F|^2(t_0,y). \qquad (5)$$

4. Einstein-Hermitian metrics

Let $\mathcal{E}$ be a reflexive sheaf on an n-dimensional compact Kähler manifold (X,ω). Then as shown in the previous section, we can solve the heat equation (2) with an admissible solution $h(t)$ untill infinite time. By Simpson [13], the stablity condition on $\mathcal{E}$ implies the existence of a subsequence $h(t_i)$, $t_i \longrightarrow \infty$ which converges to an admissible Einstein-Hermitian metric. The inequality (4) implies

$$(2c_2(\mathcal{E}) - \frac{r-1}{r}c_1(\mathcal{E})^2) \cup [\omega]^{n-2}[X] \ge c_n \int_X |F^0|^2\,\omega^n,$$

for the Einstein-Hermitian metric, which shows Corollary 3.

For a general $\mathcal{E}$, the inequality (5) enables us to take a subsequence $h(t_i)$ such that

$$\int_X |\nabla \Lambda F|^2(t_i, y) \longrightarrow 0.$$

By the results in [18], [19], taking suitable gauge change we can further take a subsequence so that the sequence of the corresponding holomorphic connections converges to a weak solution of $\nabla \Lambda F = 0$ outside a closed subset $S' \subset X \backslash S$ of locally finite Hausdorff measure of real co-dimension 4. According to the eigenspaces of ΛF, it splits into a sum of holomorphic Einstein-Hermitian vector bundles with L^2 curvature defined on $X \backslash (S \cup S')$, which extend to reflexive sheaves.

The following proposition shows the poly-stability of an Einstein-Hermitian sheaf as in the holomorphic vector bundle case.

Proposition 3. *Let $(\mathcal{E}, h)$ be a reflexive sheaf with an admissible Einstein-Hermitian metric on a compact Kähler manifold (X, ω). According as $\mu(\mathcal{E}) < 0$ or $\mu(\mathcal{E}) = 0$, the sheaf $\mathcal{E}$ admits only zero section or parallel sections $s \in \Gamma(X, \mathcal{E})$.*

Proof. If s is a global section of $\mathcal{E}$, then by Theorem 2 b), $|s|$ is bounded on X and satisfies

$$\Box |s|^2 = |\nabla s|^2 - \lambda(\mathcal{E})|s|^2 \geq 0.$$

Remark. We get the admissible Einstein-Hermitian metric solving the heat equation. It is also possible to get it as a limit of Einstein-Hermitan metrics of the holomorphic vector bundle E on X_k which is μ-stable with respect to the metrics $\omega_{k,\epsilon}$ for sufficiently small ϵ, by taking $\epsilon \longrightarrow 0$.

Acknowledgements

The first named author would like to express his thanks to Professors Itoh, Mabuchi, Noguchi and Ohsawa for helpful discussions and Professor Maruyama for pointing out an error in the first draft. He also would like to acknowledge his gratitude to Max-Planck-Institut für Mathematik for hospitality. The main part of this work was done during his stay in Bonn.

The second named author would like to acknowledge support from the National Science Foundation.

References

1. S. Bando, *Removable singularities of holomorphic vector bundles*, Tohoku Math. J. **43** (1991), 61–67.
2. S. Y. Cheng and P. Li, *Heat kernel estimates and lower bound of eigenvalues*, Comment. Math. Helv. **56** (1981), 327–338.
3. S. K. Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Diff. Geom. **18** (1983), 269–277.

4. S. K. Donaldson, *Anti self-dual Yang-Mills connections over comlpex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. **50** (1985), 1–26.
5. S. K. Donaldson, *Infinite determinants, stable bundles and curvature*, Duke Math. J. **54** (1987), 231–247.
6. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, second edition, (Springer, Berlin and New York, 1983).
7. S. Hildebrant, *Harmonic mappings of Riemannian manifolds*, in "Harmonic mappings and minimal immersions", Lect. Notes in Math. **1161**, (Springer, Berlin and New York, 1985), 1–117.
8. S. Kobayashi, *Differential geometry of holomorphic vector bundles*, Publ. Math. Soc. Japan, (Iwanami Shoten and Princeton Univ. Press, 1987).
9. J. Li and S.-T. Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, in "Mathematical aspects of string theory", Adv. Ser. Math. Phy. (World Scientific, Singapore, 1987), 560–573.
10. H. Nakajima, *Compactness of the moduli space of Yang-Mills connections in higher dimensions*, J. Math. Soc. Japan **40** (1988), 383–392.
11. T. Nitta, *Compactfication of moduli spaces of Einstein-Hermitian connections for null-correlation bundles*, in "Recent topics in differential and analytic geometry", Adv. Studies in Pure Math. **18**, (Kinokuniya, Tokyo and Academic Press Inc., Boston, New York, 1990), 397–416.
12. M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. Math. **82** (1965), 540–564.
13. C. Simpson, *Constructing variation of Hodge structure using Yang-Mills theorey and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), 867–918.
14. Y.-T. Siu, *A Hartogs type extension theorem for coherent analytic sheaves*, Ann. of Math. **93** (1971), 166–188.
15. Y.-T. Siu, *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, (Birkhäuser, Basel-Boston, 1987).
16. K. Uhlenbeck, *Removable singularities in Yang-Mills fields*, Comm. Math. Phys. **83** (1982), 11–30.
17. K. Uhlenbeck, *Connections with L^p-bounds on curvature*, Comm. Math. Phys. **83** (1982), 31–42.
18. K. Uhlenbeck, *A priori estimates for Yang-Mills fields*, preprint.
19. K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **39** (1986), S257–S293.
20. K. Uhlenbeck and S.-T. Yau, *A note on our previous paper: On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math. **42** (1989), 703–707.