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Hermitian Connections and Dirac Operators.

PAUL GAUDUCHON

...unto. – *In questo articolo si espongono i fondamenti della geometria quasi hermitiana, ed in particolare si dà una trattazione approfondita dello spazio affine delle connessioni hermitiane e dei corrispondenti operatori di Dirac. Come applicazione, si ottiene una formulazione esplicita per l'operatore di Riemann-Dirac nel generale contesto quasi hermitiano. L'ultima sezione è dedicata alle proprietà di invarianza conforme degli operatori hermitiani di Dirac.*

...troduction.

The present paper principally includes:

(i) A unified presentation of a *canonical* class of (almost) her-...itian connections, already considered by P. Libermann in [10], in-...uding the Lichnerowicz first and second canonical connections [13], ...e Libermann connection [10], the Bismut connection [2] as well as ...torsion-minimizing hermitian connection, which seems not to have ...en considered so far;

(ii) explicit expressions for the corresponding hermitian Dirac ...erators, as well as the riemannian Dirac operator in the almost-...rmitian framework, in particular for the canonical $Spin^c$-struc-...re, which are well known in the Kähler case, cf. *e.g.* [7], but seem ...be lacking in the current literature in the general case.

The first two sections of the paper constitute a reworked version ...a part of my old Lecture Notes [5] and are strongly related with ...ll-known previous contributions on the subject, in particular [10], ...], [12], [15], [13], [6]. On the other hand, no explicit mention is ...de here of curvature properties of hermitian connections; for that ...tter, the Reader may consult, in particular [13], [16], [4].
...I tried to make these Notes as self-contained as possible and I ...de no effort toward exhaustive references. As they are, I propose ...m to the Reader as a kind of a *vade mecum* for some basics of al-

most-hermitian geometry, thus partially fulfilling a whish that Franco Tricerri had often expressed to me.

Contents.

1. – The space of 2-forms with values in the tangent bundle of an almost-hermitian manifold.

1.1. *Elements of almost-hermitian geometry.* In the whole paper, (M,g,J) will denote an almost-hermitian manifold of real dimension $n=2m>2$, where: g is a (positive definite) riemannian metric, J is a g-orthogoal almost-complex structure.

We denote by: TM the (real) tangent bundle of M; T^*M the (real) cotangent bundle; $\Lambda^p M$, $p=1,\dots,n$, the bundle of real p-forms; $(\cdot,\cdot)$ the inner product induced by g on these bundles, with the following convention for the exterior and the scalar products on forms:

$$(1.1.1)\qquad \begin{aligned}(\alpha_1\wedge\dots\wedge\alpha_p)_{X_1,\dots,X_p}&=\det((\alpha_i,X_j))\,,\\ (\alpha_1\wedge\dots\wedge\alpha_p,\beta_1\wedge\dots\wedge\beta_p)&=\det((\alpha_i,\beta_j))\,,\end{aligned}$$

where det denotes the determinant. The almost-complex operator J is extended to T^*M by setting: $(J\alpha)_X=-\alpha_{JX}$, for any covector α and any vector X, so that J commutes with the riemannian duality between TM and T^*M. The Levi-Civita connection of g will be denoted by D.

We consider the following objects: the *Nijenhuis tensor*, or *complex torsion*, N, defined by:

$$(1.1.2)\qquad 4N_{X,Y}=[JX,JY]-J[JX,Y]-J[X,JY]-[X,Y]\,,$$

for any vector fields X and Y, which vanishes if and only if J is integrable (Newlander-Nirenberg theorem [14]); the *Kähler form* F, defined by: $F_{X,Y}=g_{JX,Y}$; the *Lee form* θ, defined equivalently by:

$$(1.1.3)\qquad \theta=J\delta F\,,$$

where $\delta=-*d*$ is the codifferential with respect to g, or by:

$$(1.1.4)\qquad \theta=\Lambda(dF)\,,$$

where dF denotes the exterior differential of the Kähler form F and Λ denotes the *contraction* by the Kähler form, defined, for any exterior form ψ, by: $\Lambda(\psi)=1/2\sum_{i=0}^{n}\psi(e_i,Je_i,\cdot,\dots,\cdot)$; here, and henceforth, $\{e_i\}_{i=1,\dots,n}$, denotes a *J-adapted* orthonormal frame, with $e_{2l}=Je_{2l-1}$, $l=1,\dots,m$.

The Lee form θ is also determined by:

$$(1.1.5)\qquad dF=(dF)_0+\frac{1}{(m-1)}\,\theta\wedge F\,,$$

where $(dF)_0$ denotes the *primitive* part of dF, *i.e.* its orthogonal projection of into the kernel of Λ.

Notice that, for $n=4$, (1.1.5) is reduced to: $dF=\theta\wedge F$.

In addition to dF, we will consider the real 3-form d^cF defined by:

$$(1.1.6)\qquad (d^cF)_{X,Y,Z}=-dF_{JX,JX,JZ}\,,$$

which plays an important role in the sequel.

The almost-hermitian structure (g,J) is said to be: *Kähler* if J is D-parallel; *symplectic* if F is closed; $(2,1)$-*symplectic* if $(dF)^+ \equiv 0$, where $(dF)^+$ denotes the part of type $(2,1)+(1,2)$ of dF; *hermitian* if J is integrable, cf. Remark 5 below. As it is well-known, (g,J) is Kähler if and only if it is both $(2,1)$-symplectic and hermitian, cf. *e.g.* [8].

1.2. *The space of TM-valued 2-forms.* We denote by $\Omega^2(TM)$ the space of TM-valued 2-forms on M, *i.e.* the space of sections of the tensor product $TM \otimes \Lambda^2 M$.

Each element B de $\Omega^2(TM)$ will also be considered (via g) as a real trilinear form, skew-symmetric with respect to the two *last* arguments, by setting: $B_{X,Y,Z} = (X, B_{Y,Z})$.

In particular, the space $\Omega^3 M$ of (real) 3-forms will be considered as a sub-space of $\Omega^2(TM)$. We denote by $\mathfrak{b}$ the *Bianchi projector* of $\Omega^2(TM)$ onto $\Omega^3 M$, defined by:

$$(\mathfrak{b}B)_{X,Y,Z} = \frac{1}{3}(B_{X,Y,Z} + B_{Y,Z,X} + B_{Z,X,Y}). \tag{1.2.1}$$

The *trace* of an element B of $\Omega^2(TM)$ is the real 1-form $\mathrm{tr}(B)$ defined by:

$$\mathrm{tr}(B)_X = \sum_{i=1}^{n} B_{e_i,e_i,X}. \tag{1.2.2}$$

The trace can be considered as a projector of $\Omega^2(TM)$ onto the space $\Omega^1 M$ of real 1-forms, realized as a subspace of $\Omega^2(TM)$ as follows: identify any (real) 1-form α with the element $\tilde{\alpha}$ of $\Omega^2(TM)$ defined by:

$$\tilde{\alpha}_{X,Y,Z} = \frac{1}{(n-1)}(\alpha_Z g_{X,Y} - \alpha_Y g_{Z,X}), \tag{1.2.3}$$

so that: $\mathrm{tr}\,\tilde{\alpha} = \alpha$.

We thus get the following orthogonal decomposition of $\Omega^2(TM)$:

$$\Omega^2(TM) = \Omega^1 M \oplus (\Omega^2(TM))^0 \oplus \Omega^3 M, \tag{1.2.4}$$

where $(\Omega^2(TM))^0$ denotes the sub-space of trace-free elements of $\Omega^2(TM)$ satisfying the Bianchi identity. Accordingly, any element B of $\Omega^2 TM$ can be written as follows:

$$B = \widetilde{\mathrm{tr} B} + B^0 + \mathfrak{b}B, \tag{1.2.5}$$

where B^0 is trace-free and satisfies the *Bianchi identity*: $\mathfrak{b}B^0 = 0$.

1.3. *The type of a TM-valued 2-form.* The decomposition (1.2.4) of the space $\Omega^2(TM)$ follows from the corresponding decomposition of the vector bundle $TM \otimes \Lambda^2 M$, while the latter refers to the decomposition of the space $R^n \otimes \Lambda^2(R^n)^*$ under the action of the orthogonal group $O(n)$. The next step consists in refining this decomposition when restricting the action of $O(n)$ to the unitary group $U(m)$. For that, we introduce the following notation:

DEFINITION 1. – *An element B of $\Omega^2(TM)$, viewed as a TM-valued 2-form, is said to be:*

of type $(1,1)$, *if* $B_{JX,JY} = B_{X,Y}$,
of type $(2,0)$, *if* $B_{JX,Y} = J(B_{X,Y})$,
of type $(0,2)$, *if* $B_{JX,Y} = -J(B_{X,Y})$,

for any vector fields X, Y.

We denote by $\Omega^{0,2}(TM)$, $\Omega^{2,0}(TM)$ and $\Omega^{1,1}(TM)$ the subspaces of $\Omega^2(TM)$ of elements of type $(0,2)$, $(2,0)$ and $(1,1)$ respectively, and, for each element B of $\Omega^2(TM)$, we denote by $B^{0,2}$, $B^{2,0}$ and $B^{1,1}$ the corresponding components. For later use, we introduce the involution $\mathfrak{M}$ of $\Omega^2(TM)$ defined by:

$$(\mathfrak{M}B)_{X,Y,Z} = B_{X,JY,JZ}. \tag{1.3.1}$$

This involution is an isometry. Its eigenspaces with respect to $+1$ and -1 are the subspaces $\Omega^{1,1}(TM)$ and $\Omega^{0,2}(TM) \oplus \Omega^{2,0}(TM)$ respectively.

REMARK 1. – The terminology of Definition 1 is motivated by the fact that the *real* tangent bundle TM of M can also be considered as a *complex* vector bundle of rank m by identifying the action of J with the multiplication by the complex number $i = \sqrt{-1}$. Nevertheless, elements of $\Omega^2(TM)$ are *real*; in particular, the subspace $\Omega^{0,2}(TM)$ is *not* the «conjugate» of $\Omega^{2,0}(TM)$ in any sense.

In terms of representations of the unitary group $U(m)$, the subspaces $\Omega^{0,2}(TM)$, $\Omega^{2,0}(TM)$ and $\Omega^{1,1}(TM)$ correspond to the real parts of the complex representations $V \otimes \Lambda^2 V$, $\bar{V} \otimes \Lambda^2 V$ and $\bar{V} \otimes V \otimes V$ respectively, where $V = \mathbb{C}^m$ denotes the standard representation of $U(m)$. In particular, the (real) dimensions of these spaces are $m^2(m-1)$, $m^2(m-1)$ and $2m^3$ respectively.

REMARK 2. – As observed above, a (real) 3-form ψ on M can be viewed as a (totally skew-symmetric) section of $\Omega^2(TM)$. It thus admits *two* different type decompositions:

(i) a decomposition as a 3-form:

$$\psi = \psi^- + \psi^+ , \tag{1.3.2}$$

where ψ^+ denotes the $((2,1)+(1,2))$-part and ψ^- the $((3,0)+(0,3))$-part of ψ in the usual sense;

(ii) a type decomposition as an element of $\Omega^2(TM)$, according to the Definition 1. The two decompositions are related by:

$$\psi^- = \psi^{0,2} , \tag{1.3.3}$$

$$\psi^+ = \psi^{2,0} + \psi^{1,1} , \tag{1.3.4}$$

$$\psi^{2,0} = \frac{1}{2}(\psi^+ - \mathcal{M}\psi^+), \qquad \psi^{1,1} = \frac{1}{2}(\psi^+ + \mathcal{M}\psi^+). \tag{1.3.5}$$

On the other hand, any 3-form ψ^+ of type $(2,1)+(1,2)$ (as a 3-form), satisfies the identity:

$$\psi^+(X,Y,Z) = \psi^+(X,JY,JZ) + \psi^+(JX,Y,JZ) + \psi^+(JX,JY,Z), \tag{1.3.6}$$

or, equivalently:

$$\psi^+ = 3\mathfrak{b}(\mathcal{M}\psi^+). \tag{1.3.7}$$

Since $\psi^+ = \mathfrak{b}\psi^+$, the above identity immediately implies:

$$(\psi^+, \mathcal{M}\psi^+) = \frac{1}{3}|\psi^+|^2 . \tag{1.3.8}$$

It then follows from (1.3.5) that the norms of the componemts $\psi^{2,0}$ and $\psi^{1,1}$ of ψ are related by:

$$|\psi^{2,0}|^2 = \frac{1}{2}|\psi^{1,1}|^2 . \tag{1.3.9}$$

In particular, a non-zero 3-form ψ^+ of type $(2,1)+(1,2)$ (as a 3-form) cannot be of «pure type» as an element of $\Omega^2(TM)$.

In the sequel, the spaces of real 3-forms of type $(2,1)+(1,2)$ and $(3,0)+(0,3)$ will be denoted by $\Omega^{3,+}M$ and $\Omega^{3,-}M$ respectively.

Notice that $\Omega^{3,-}M$ is contained in the space $\Omega^3_0 M$ of primitive 3-forms.

For $n=4$, the spaces $\Omega^{3,-}M$ and $\Omega^3_0 M$ are both reduced to $\{0\}$.

1.4. *Symmetries of elements of pure type in $\Omega^2(TM)$.* The relationship between (1.2.4) and the decomposition into types of elements of $\Omega^2(TM)$ is explained in the next three lemmas.

LEMME 1. – *For any element B of $\Omega^{0,2}$, we have:*

(i) $\mathrm{tr}(B) = 0$;

(ii) *the components B^0 and $\mathfrak{b}B$ both belong to $\Omega^{0,2}(TM)$.*

In particular, as a 3-form, $\mathfrak{b}B$ is of type $(3,0)+(0,3)$.

LEMME 2. – *The restriction of the Bianchi projector $\mathfrak{b}$ to $\Omega^{2,0}(TM)$ is an isomorphism from $\Omega^{2,0}(TM)$ onto $\Omega^{3,+}M$.*

More precisely, for any B in $\Omega^2(TM)$, $\mathfrak{b}B$ belongs to $\Omega^{3,+}M$ and we have:

$$B = \frac{3}{2}(\mathfrak{b}B - \mathcal{M}(\mathfrak{b}B)) . \tag{1.4.1}$$

LEMME 3. – (i) *The space $\Omega^{1,1}(TM)$ admits the following orthogonal splitting:*

$$\Omega^{1,1}(TM) = \Omega^{1,1}_s(TM) \oplus \Omega^{1,1}_a(TM), \tag{1.4.2}$$

where $\Omega^{1,1}_s(TM)$ denotes the subspace of elements of $\Omega^{1,1}(TM)$ satisfying the Bianchi identity: $\mathfrak{b}B = 0$, and $\Omega^{1,1}_a(TM)$ denotes the subspace of $\Omega^{1,1}(TM)$ orthogonal to $\Omega^{1,1}_s(TM)$.

(ii) *The restriction of $\mathfrak{b}$ to $\Omega^{1,1}_a(TM)$ is an isomorphism from $\Omega^{1,1}_a(TM)$ onto $\Omega^{3,+}M$.*

More precisely, for any A in $\Omega^{1,1}_a(TM)$, $\mathfrak{b}A$ belongs to $\Omega^{3,+}M$ and we have:

$$A = \frac{3}{4}(\mathfrak{b}A + \mathcal{M}(\mathfrak{b}A)) . \tag{1.4.3}$$

PROOF. – For any B in $\Omega^{0,2}$, we have:

$$B_{JX,Y,Z} = B_{X,JY,Z} = B_{X,Y,JZ} . \tag{1.4.4}$$

Lemma 1 follows easily. Lemmas 2 and 3 can be deduced from the Schur Lemma or by direct verification. For example, the injectivity of the restriction of $\mathfrak{b}$ to $\Omega^{2,0}(TM)$ can be directly checked as follows:

if B in $\Omega^{2,0}(TM)$ satisfies the Bianchi identity: $\mathfrak{b}B=0$, we have (by applying the Bianchi identity two times):

$$0=B_{X,JY,JZ}+B_{JY,JZ,X}+B_{JZ,X,JY}=$$
$$-B_{X,Y,Z}+B_{Y,Z,X}+B_{Z,X,Y}=-2B_{X,Y,Z}. \quad \blacksquare$$

REMARK 3. – The subspaces $\Omega^{1,1}_s(TM)$ and $\Omega^{1,1}_a(TM)$ are respectively associated with the representation spaces $\bar{V}\otimes S^2V$ and $\bar{V}\otimes\Lambda^2V$ of $U(m)$, where S^2V, resp. Λ^2V, denotes the symmetric, resp. skew-symmetric, part of $V\otimes V$. By Lemmas 2 and 3, the Bianchi projector $\mathfrak{b}$ identifies each space $\Omega^{2,0}(TM)$ and $\Omega^{1,1}_a(TM)$ with the space $\Omega^{3,+}M$ of 3-forms of type $(2,1)+(1,2)$. We infer an isomorphism from $\Omega^{2,0}(TM)$ onto $\Omega^{1,1}_a(TM)$, denoted by Φ, and the inverse isomorphism from $\Omega^{1,1}_a(TM)$ onto $\Omega^{2,0}(TM)$, denoted by Φ^{-1}, given by:

(1.4.5) $$\Phi(B)=\frac{3}{4}(\mathfrak{b}B+\mathfrak{M}(\mathfrak{b}B)), \quad \forall B\in\Omega^{2,0}(TM),$$

(1.4.6) $$\Phi^{-1}(A)=\frac{3}{2}(\mathfrak{b}A-\mathfrak{M}(\mathfrak{b}A)), \quad \forall A\in\Omega^{1,1}_a(TM)$$

(direct consequence of (1.4.1)-(1.4.3)). The isomorphisms Φ and Φ^{-1} are not isometries; more precisely, by (1.3.8), we have:

(1.4.7) $$|\Phi(B)|^2=\frac{1}{2}|B|^2, \quad \forall B\in\Omega^{2,0}(TM).$$

It may be observed that Φ is also given by:

(1.4.8) $$(\Phi(B))_{X,Y,Z}=\frac{1}{2}(B_{Y,Z,X}+B_{Z,X,Y}), \quad \forall B\in\Omega^{2,0}(TM).$$

REMARK 4. – By (1.4.1) and (1.4.3), for any B in $\Omega^{2,0}(TM)$ and any A in $\Omega^{1,1}_a(TM)$, we have:

(1.4.9) $$\mathrm{tr}(B)=3J\Lambda(\mathfrak{b}B), \quad \mathrm{tr}(A)=-\frac{3}{2}J\Lambda(\mathfrak{b}A).$$

2. – Canonical hermitian connections on an almost-hermitian manifold.

2.1. *Potential and torsion of a hermitian connection.* For any almost-hermitian manifold (M,g,J), a linear connection ∇ on M (acting on sections of the tangent bundle TM) is *hermitian* if it preserves the metric g and the almost-complex structure J: $\nabla g=0$ and $\nabla J=0$ (the name *hermitian* for connections does'nt suppose J being integrable).

We denote by A^∇, or simply A, the *potential* of ∇ (with respect to the Levi-Civita connection D), defined as the difference $\nabla-D$. Since ∇ and D both preserve the metric g, the potential A can, and will, be considered as an element of $\Omega^2(TM)$ by setting:

(2.1.1) $$A_{X,Y,Z}=(\nabla_XY,Z)-(D_XY,Z).$$

The space of hermitian connections on M is an affine space, denoted by $\mathcal{C}(g,J)$, modelled on the subspace $\Omega^{1,1}(TM)$ of $\Omega^2(TM)$.

Since preserving g, ∇ is entirely determined by its *torsion* T^∇, or simply T. In the sequel, we will always consider T as an element of $\Omega^2(TM)$ by setting:

(2.1.2) $$T_{X,Y,Z}=(X,T_{Y,Z}).$$

The potential $A=A^\nabla$ and the torsion $T=T^\nabla$ of any hermitian connection ∇ are related by:

(2.1.3) $$T=-A+3\mathfrak{b}A,$$

(2.1.4) $$A=-T+\frac{3}{2}\mathfrak{b}T.$$

In particular, we have:

(2.1.5) $$\mathfrak{b}A=\frac{1}{2}\mathfrak{b}T, \quad \mathrm{tr}(A)=-\mathrm{tr}(T), \quad A^0=-T^0.$$

2.2. *Structure of DF.* In addition to the potential A and the torsion T of any hermitian connection ∇, the following tensors will also be considered as elements of $\Omega^2(TM)$: the Nijenhuis tensor N of J, by setting:

(2.2.1) $$N_{X,Y,Z}=(X,N_{Y,Z});$$

the covariant derivative DF of the Kähler form F with respect to the Levi-Civita connection D, by setting:

(2.2.2) $$(DF)_{X,Y,Z}=(D_XF)_{Y,Z}.$$

Then we have (cf. also [6]):

PROPOSITION 1. – (i) *The Nijenhuis tensor N is of type $(0,2)$; in* *particular, it is trace-free and splits as follows*:

$$N = N^0 + \mathfrak{b}N\,, \tag{2.2.3}$$

where N^0 *satisfies the Bianchi identity and* $\mathfrak{b}N$ *is the skew-symmetric part of* N. *Moreover,* $\mathfrak{b}N$ *is entirely determined by the component* $(dF)^-$ *of* dF; *more precisely, we have*:

$$\mathfrak{b}N = \frac{1}{3}(d^cF)^-\,. \tag{2.2.4}$$

(ii) *The component* $(DF)^{1,1}$ *of* DF *vanishes identically, i.e. the type decomposition of* DF *is reduced to*:

$$DF = (DF)^{0,2} + (DF)^{2,0}\,. \tag{2.2.5}$$

(iii) *The component* $(DF)^{0,2}$ *of* DF *is entirely determined by and determines* N; *more precisely, we have*:

$$(DF)^{2,0}_{X,Y,Z} = 2N^0_{JX,Y,Z} + \frac{1}{3}(dF)^-_{X,Y,Z} = 2N_{JX,Y,Z} + (dF)^-_{X,Y,Z}\,, \tag{2.2.6}$$

or, equivalently:

$$(DF)^{0,2}_{X,Y,Z} = N_{JX,Y,Z} + N_{JY,X,Z} - N_{JZ,X,Y}\,. \tag{2.2.7}$$

In particular, we have the equivalence:

$$D_{J.}J = J\circ D.J \Leftrightarrow J \text{ is integrable}\,. \tag{2.2.8}$$

(iv) *The component* $(DF)^{2,0}$ *of* DF *is entirely determined by* $(dF)^+$; *more precisely, we have*:

$$(DF)^{2,0}_{X,Y,Z} = \frac{1}{2}\left((dF)^+_{X,Y,Z} - (dF)^+_{X,JY,JZ}\right). \tag{2.2.9}$$

In particular, we have the following equivalence:

$$D_{J.}J = -J\circ D.J \Leftrightarrow (g,J) \text{ is } (2,1)\text{-symplectic}\,. \tag{2.2.10}$$

PROOF. – The fact that N is of type $(0,2)$, as an element of $\Omega^2(TM)$, is an immediate consequence of its definition. The identity (2.2.7) can be checked directly; (2.2.4) and (2.2.6) follow directly. The identity (2.2.9) is a consequence of (1.4.1) and the fact that $\mathfrak{b}(DF)$ is equal to $(1/3)\,dF$. The component $(DF)^{1,1}$ being equal to zero is an immediate consequence of the fact that DJ anticommutes with J. Finally, as an element of $\Omega^2(TM)$, DF is of type $(2,0)$ if and only if $D_{JX} = J\circ D_XJ$ and of type $(0,2)$ if and only if $D_{JX}J = -J\circ D_XJ$, for any vector field X. ■

COROLLARY 1. – *For any almost-hermitian structure* (g,J), *we have the inequality*:

$$|DF|^2 = |dF|^2 + 4|N^0|^2 - \frac{2}{3}|(dF)^-|^2\,. \tag{2.2.11}$$

In particular, if J *is integrable, we have*: $|DF| = |dF|$.
If $n=4$, *we have the inequality*:

$$|DF|^2 \geq |dF|^2\,, \tag{2.2.12}$$

with equality if and only if J *is integrable.*

PROOF. – Immediate consequence of (2.2.6) and (2.2.9). If $n=4$, the component $(dF)^-$ vanishes identically for any almost-hermitian structure. ■

NOTE. – The norms appearing in the Corollary above are *not* the tensorial norms; instead, the norms for dF or $(dF)^-$ is the norms in Λ^3M, according to (1.1.1), whereas the norm for DF or N^0 is the norm in $TM\otimes\Lambda^2M$, with the same convention for the factor Λ^2M.

REMARK 5. – By Proposition 1, for a general almost-hermitian structure (g,J), DF is the sum of *four* components, each corresponding to some irreducible representation of $U(m)$, respectively identified to N^0, $(dF)^-$, θ and $(dF)^+_0$.

The celebrated *Gray-Hervella classification* directly follows from this decomposition: each of the $16=2^4$ categories of almost-hermitian structures listed in [6] corresponds to the vanishing of some subset of the set of 4 «irreducible components» of DF. In this paper, in addition to the *general* case (no condition) and the *Kähler* case (all components vanish), we only consider the *symplectic* or *almost Kähler* case (vanishing of the three components $(dF)^-$, θ and $(dF)^+_0$), the $(2,1)$-*symplectic* case (vanishing of the two components θ and $(dF)^+_0$) and the *hermitian* or *integrable* case (vanishing of $(dF)^-$ and N^0). Notice that the two latter categories are complementary to each other, cf. also (2.2.8) and (2.2.10) in Proposition 1) (the terminology concerning the various categories of almost-hermitian structures has not been definitively settled yet; the name $(2,1)$-*symplectic, quasi-Kähler* in [6], has been proposed by S. Salamon).

For $n=4$, the Gray-Hervella classification is reduced to $4=2^2$ categories; in particular, in this case the symplectic and $(2,1)$-symplectic categories coincide.

2.3. *Describing a hermitian connection by its torsion.* As we already observed, a hermitian connection ∇ is entirely determined by its torsion T. The properties of T are described by the following proposition, cf. also [13], [12]:

PROPOSITION 2. – *For any hermitian connection ∇, let T be the torsion of ∇, viewed as an element of $\Omega^2(TM)$. Then:*

(i) *The component $T^{0,2}$ of type $(0,2)$ of T is independent of ∇, equal to the Nijenhuis tensor N:*

$$T^{0,2} = N. \tag{2.3.1}$$

(ii) *The skew-symmetric part of $(T^{2,0} - T_a^{1,1})$ is independent of ∇, equal to $(1/3)(d^cF)^+$:*

$$\mathfrak{b}(T^{2,0} - T_a^{1,1}) = \frac{1}{3}(d^cF)^+ . \tag{2.3.2}$$

Equivalently, the difference $(T^{2,0} - \Phi^{-1}(T_a^{1,1}))$, in $\Omega^{2,0}TM$, is independent of ∇. equal to:

$$T^{2,0} - \Phi^{-1}(T_a^{1,1}) = \frac{1}{2}((d^cF)^+ - \mathfrak{M}(d^cF)^+) = (DF)^{2,0}(J.,\cdot,\cdot). \tag{2.3.3}$$

(iii) *T is entirely determined by its component $T_s^{1,1}$ and its component $(\mathfrak{b}T)^+$, which can be chosen arbitrarily.*

More precisely, for any real 3-form ψ^+ of type $(2,1)+(1,2)$ and any section B_s of $\Omega_s^{1,1}(TM)$, there exists a unique hermitian connection ∇, whose torsion T satisfies the two following conditions:

$$T_s^{1,1} = B_s, \qquad (\mathfrak{b}T)^+ = \psi^+ . \tag{2.3.4}$$

The remaining part of the torsion is then determined by (2.3.1) and by:

$$\begin{aligned} \mathfrak{b}(T_a^{2,0}) &= \frac{1}{2}\left(\psi^+ + \frac{1}{3}(d^cF)^+\right), \\ \mathfrak{b}(T_a^{1,1}) &= \frac{1}{2}\left(\psi^+ - \frac{1}{3}(d^cF)^+\right), \end{aligned} \tag{2.3.5}$$

i.e the torsion is given by:

$$T = N + \frac{1}{8}(d^cF)^+ - \frac{3}{8}\mathfrak{M}(d^cF)^+ + \frac{9}{8}\psi^+ - \frac{3}{8}\mathfrak{M}\psi^+ + B_s . \tag{2.3.6}$$

The corresponding hermitian connection ∇ is then equal to $D+A$. where the potential A is obtained from T by (2.1.4).

PROOF. – The linear connection $\nabla = D + A$ is hermitian if and only if A, as a section of $\Omega^2(TM)$, satisfies

$$A_{X,JY,Z} + A_{X,Y,JZ} = -(DF)_{X,Y,Z} . \tag{2.3.7}$$

Due to (2.1.4), this holds if and only if the torsion T of ∇ satisfies:

$$T_{X,JY,Z} + T_{X,Y,JZ} - \frac{3}{2}((\mathfrak{b}T)_{X,JY,Z} + (\mathfrak{b}T)_{X,Y,JZ}) = (DF)_{X,Y,Z} . \tag{2.3.8}$$

Since $T^{1,1}_{X,JY,Z} + T^{1,1}_{X,Y,JZ} \equiv 0$, the condition (2.3.8) is equivalent to the following system:

$$2T^{0,2}_{JX,Y,Z} - 3(\mathfrak{b}T)^-_{JX,Y,Z} = DF^{0,2}_{X,Y,Z} , \tag{2.3.9}$$

$$-2T^{2,0}_{JX,Y,Z} - \frac{3}{2}((\mathfrak{b}T)^+_{X,JY,Z} + (\mathfrak{b}T)^+_{X,Y,JZ}) = DF^{2,0}_{X,Y,Z} . \tag{2.3.10}$$

Then, assertion (i) follows easily from (2.3.9) and (2.2.6).

By (1.4.1) and (1.4.3), (2.3.10) is easily checked to be equivalent to (2.3.3) (itself equivalent to (2.3.2)). This proves (ii).

Finally, the above argument shows that ∇ is hermitian if and only if its torsion T satisfies (2.3.1) and (2.3.2). This proves (iii). ■

2.4. *Hermitian connections of minimal torsion.* For reasons which will become clear in the sequel, the 3-form ψ^+ that appears in (2.3.5)-(2.3.6) is better expressed as follows:

$$\psi^+ = \frac{(2f-1)}{3}(d^cF)^+ + \psi_0^+ , \tag{2.4.1}$$

where f is a real function on M and ψ_0^+ is orthogonal to $(d^cF)^+$ at any point of M.

Then, (2.3.6) reads as follows:

$$T = N + \frac{(3f-1)}{4}(d^cF)^+ - \frac{(f+1)}{4}\mathfrak{M}(d^cF)^+ + \frac{9}{8}\psi_0^+ - \frac{3}{8}\mathfrak{M}\psi_0^+ + B_s . \tag{2.4.2}$$

From (2.4.2), we easily infer

PROPOSITION 3. – *For any hermitian connection ∇, the torsion T*

of ∇ satisfies the following inequality, at any point of M:

$$|T|^2 \geqslant |N^0|^2 + \frac{1}{3}|dF|^2 . \tag{2.4.3}$$

Equality in (2.4.3) *holds if and only if T satisfies:*

$$T_s^{1,1} = 0 , \qquad (\mathfrak{b}T)^+ = -\frac{1}{9}(d^cF)^+ , \tag{2.4.4}$$

if and only if T satisfies:

$$T = N - \frac{1}{3}\mathfrak{M}(d^cF)^+ . \tag{2.4.5}$$

Proof. – From (2.4.2), (2.2.4) and (1.3.8), we infer the following series of equalities and inequalities:

$$\begin{aligned} &|T|^2 = \quad (2.4.6)\\ &|N^0|^2 + \frac{1}{3}|(d^cF)^-|^2 + |B_s|^2 + \frac{1}{2}(3f^2 - 2f + 1)|(d^cF)^+|^2 + \frac{27}{8}|\psi_0^+|^2 \geqslant \\ &|N^0|^2 + \frac{1}{3}|(d^cF)^-|^2 + \frac{1}{2}(3f^2 - 2f + 1)|(d^cF)^+|^2 \geqslant \\ &|N^0|^2 + \frac{1}{3}|(d^cF)^-|^2 + \frac{1}{3}|(d^cF)^+|^2 = \\ &|N^0|^2 + \frac{1}{3}|dF|^2 . \end{aligned}$$

Equality holds at any point if and only if B_s and ψ_0^+ both vanish identically and the function f is constant, equal to $1/3$. ∎

Corollary 2. – *For any almost-hermitian structure (g, J) on M, there exists a uniquely defined hermitian connection, denoted by $\nabla^{\min}$, characterized by its torsion being of minimal norm at any point of M.*

This connection is related to the Levi-Civita connection D by:

$$(\nabla_X^{\min} Y, Z) = (D_X Y, Z) - (X, N_{Y,Z}) + \frac{1}{2}(d^cF)_{X,Y,Z} - \frac{1}{6}(d^cF)^+_{X,Y,Z} + \frac{1}{3}(d^cF)^+_{X,JY,JZ} . \tag{2.4.7}$$

Proof. – The first part is a direct consequence of the above proposition; (2.4.7) follows from (2.1.4) and (2.4.5). ∎

2.5. *The set of canonical hermitian connections.* Proposition 2 (ii) singles out a distinguished set of hermitian connections (a real affine line or a point in $\mathcal{A}(g, J)$), namely the *canonical* hermitian connections defined as follows (cf. also [10]):

Definition 2. – *For any almost-hermitian structure (g, J), a hermitian connection is called* canonical *if its torsion T satisfies the following condition:*

$$T_s^{1,1} = 0 , \qquad (\mathfrak{b}T)^+ = \frac{(2t-1)}{3}(d^cF)^+ . \tag{2.5.1}$$

for some real number t.

The canonical hermitian connection corresponding to the value t of the parameter will be denoted by ∇^t.Then, by (2.4.2), the torsion T^t of ∇^t is expressed as:

$$T^t = N + \frac{1}{4}(3t-1)(d^cF)^+ - \frac{1}{4}(t+1)\mathfrak{M}(d^cF)^+ , \tag{2.5.2}$$

while, by (2.1.4), the connection ∇^t itself is related to the Levi-Civita connection D by:

$$\begin{aligned} (\nabla_X^t Y, Z) = (D_X Y, Z) - (X, N_{Y,Z}) + &\quad (2.5.3)\\ \frac{1}{2}(d^cF)^-_{X,Y,Z} - \frac{1}{4}(d^cF)^+_{X,Y,Z} + \frac{1}{4}(d^cF)^+_{X,JY,JZ} + &\\ \frac{t}{4}((d^cF)^+_{X,Y,Z} + (d^cF)^+_{X,JY,JZ}) . & \end{aligned}$$

By (2.2.6), (2.5.3) also reads as follows:

$$(\nabla_X^t Y, Z) = (D_X Y, Z) + \frac{1}{2}((D_X J)JY, Z) + \frac{t}{4}((d^cF)^+_{X,Y,Z} + (d^cF)^+_{X,JY,JZ}) . \tag{2.5.4}$$

In particular, the hermitian connection ∇^0 coincides with the orthogonal projection of the Levi-Civita connection D into the affine space $\mathcal{A}(g, J)$, in the following sense: by considering the Levi-Civita connection D as the natural origin of the affine space $\mathcal{A}(g)$ of connections on M that preserve g, $\mathcal{A}(g)$ is identified with the space $\Omega^2(TM)$ of sections of the fiber bundle $TM \otimes \Lambda^2 M$, while $\mathcal{A}(g, J)$ is identified with the space of sections of the *affine* sub-bundle, say $(TM \otimes \Lambda^2 M)^J$, of elements A satisfying (2.3.7); then, at any point of

M, ∇^0 is equal to the orthogonal projection of the origin into the fiber of $(TM \otimes \Lambda^2 M)^J$ at this point.

Alternatively, let ω denote the connection 1-form of the Levi-Civita connection D with respect to some J-adapted orthonormal frame $e = \{e_1, e_2 = Je_1, \ldots, e_{2m-1}, e_{2m} = Je_{2m-1}\}$ of TM; then, the connection 1-form $\underset{\mathfrak{so}(2m)}{\omega^0}$ of $\underset{\mathfrak{so}(2m)}{\nabla^0}$ with respect to e is given by: $\omega^0(X) = \prod_{\mathfrak{u}(m)} (\omega(X))$, where, $\prod_{\mathfrak{u}(m)}$ denotes the natural projection from $\mathfrak{so}(2m)$ onto the sub-Lie algebra $\mathfrak{u}(m)$ of $U(m)$. Hence, ∇^0 coincides with the *first canonical connection* of [13]. This fact is a justification *a posteriori* of the choice of the parameter t in (2.5.1).

PROPOSITION 4. – *For any almost-hermitian structure, each canonical hermitian connection ∇^t is related to the «origin» ∇^0 by:*

(2.5.5)
$$\nabla^t = \nabla^0 + \frac{t}{4}((d^c F)^+ + \mathfrak{M}(d^c F)^+) .$$

In particular, if (g,J) is $(2,1)$-symplectic, the canonical set $\{\nabla^t\}_{t \in \mathbb{R}}$ is reduced to $\{\nabla^0\}$, while, if (g,J) is not $(2,1)$-symplectic, elements of $\{\nabla^t\}_{t\in\mathbb{R}}$ corresponding to different values of t are distinct and the set of canonical hermitian connections form an affine line in $\mathfrak{A}(g,J)$:

(2.5.6)
$$\nabla^t = t\nabla^1 + (1-t)\nabla^0 ,$$

where ∇^0 and ∇^1 are distinct, respectively identified with the first and the second canonical connection of [13].

PROOF. – Immediate consequence of (2.5.3). Cf. also [10] and the next subsection. ■

For the convenience of the Reader, we recall the expression of the components of the torsion T^t of ∇^t (directly deduced from (2.5.2)):

(2.5.7)
$$(T^t)^{2,0} = \frac{t}{2}((d^c F)^+ - \mathfrak{M}(d^c F)^+) ,$$

(2.5.8)
$$(T^t)^{1,1}_a = \frac{(t-1)}{4}((d^c F)^+ + \mathfrak{M}(d^c F)^+) ,$$

(2.5.9)
$$(T^t)^{1,1}_s = 0 ,$$

(2.5.10)
$$\mathfrak{b}T^t = \frac{(2t-1)}{3}(d^c F)^+ + \frac{1}{3}(d^c F)^- ,$$

(2.5.11)
$$\operatorname{tr}(T^t) = -\frac{(t+1)}{2}\theta , \qquad \operatorname{tr}((T^t)^{1,1}) = \frac{(t-1)}{2}\theta , \qquad \operatorname{tr}((T^t)^{2,0}) = -t\theta .$$

2.6. *Some distinguished canonical hermitian connections.* Within the canonical set $\{\nabla^t\}_{t\in\mathbb{R}}$ of hermitian connections, we distinguish the following ones:

$t=0$; as already noticed, ∇^0 is the *first canonical connection* of [13], *i.e.* the orthogonal projection of the Levi-Civita connection D into the affine space $\mathfrak{A}(g,J)$ as explained above. This connection is characterized in $\mathfrak{A}(g,J)$ by the condition:

(2.6.1)
$$T^{1,1}_s = 0 , \quad T^{2,0} = 0 .$$

In particular, if J is integrable, the torsion of ∇^0 is of type $(1,1)$.

$t=1$; the connection ∇^1 is the *second canonical connection* of [13]; it is also called the *Chern connection* of (g,J) since, in the integrable case, it coincides with the *Chern connection* of the tangent bundle viewed as a hermitian, holomorphic bundle of rank m, cf. [3].

It is characterized, in $\mathfrak{A}(g,J)$, by the condition:

(2.6.2)
$$T^{1,1} = 0 .$$

Equivalently, ∇^1 is characterized in $\mathfrak{A}(g,J)$ by its part of type $(0,1)$ coinciding with the Cauchy-Riemann operator $\bar\partial$, acting on sections of TM, defined by:

(2.6.3)
$$\bar\partial_X Y = -\frac{1}{2}J(\mathcal{L}_Y J)(X) + N_{X,Y} \quad (\text{i.e. } N_{-\{X,Y\}}) ,$$

where $\mathcal{L}$ denotes the Lie derivative, cf. Proposition 4 below.

$t=-1$; ∇^{-1} has been considered by J.-M. Bismut in [2]. It is characterized in $\mathfrak{A}(g,J)$ by the condition:

(2.6.4)
$$T - N \quad \text{is skew-symmetric} .$$

In particular, if J is integrable, ∇^{-1} is characterized by its torsion being skew-symmetric.

In the canonical set $\{\nabla^t\}_{t\in\mathbf{R}}$, ∇^{-1} is characterized by the associated Dirac operator P^{-1} being *self-dual* [2], cf. also Proposition 6 below and [2] for further developments.

$t=1/2$; $\nabla^{1/2}$ has been called *conformal* by P. Libermann in [10]. It is characterized in $\mathcal{A}(g,J)$ by the condition:

$$(2.6.5) \qquad T-N \quad \text{satisfies the Bianchi identity}.$$

In the canonical set $\{\nabla^t\}_{t\in\mathbf{R}}$, $\nabla^{1/2}$ is characterized by the corresponding Dirac operator $P^{1/2}$ being *conformally covariant*, cf. [10] and Proposition 9 below.

$t=1/3$; $\nabla^{1/3}=\nabla^{\min}$ is characterized in $\mathcal{A}(g,J)$ by its torsion being minimal, cf. Corollary 2 above.

2.7. *Hermitian connections and Cauchy-Riemann operators.* As already observed, any almost-complex structure J determines a *pseudo-holomorphic structure* on the tangent space TM defined by (2.6.3) or by:

$$(2.7.1) \qquad \bar\partial_X Y=\frac{1}{4}([X,Y]+[JX,JY]+J[JX,Y]-J[X,JY]),$$

for any real vector fields X and Y, or else by:

$$(2.7.2) \qquad \bar\partial_X Y=[X,Y]^{1,0},$$

for any *complex* vector fields X and Y, of type $(0,1)$ and $(1,0)$ respectively. The *Cauchy-Riemann operator* so defined is independent of the metric and will be called the *intrinsic* Cauchy-Riemann operator of J.

On the other hand, each J-linear connection ∇ also determines a «Cauchy-Riemann operator», denoted by $\bar\partial^\nabla$, defined as the $(0,1)$-part of ∇:

$$(2.7.3) \qquad \bar\partial^\nabla_X Y=\frac{1}{2}(\nabla_X Y+J\nabla_{JX}Y).$$

In particular, a «Cauchy-Riemann operator», denoted by $\bar\partial^t$, is attached in this manner to each canonical hermitian connection ∇^t.

PROPOSITION 5. – *For any almost-hermitian structure and any canonical hermitian connection ∇^t, the corresponding Cauchy-Riemann operator $\bar\partial^t$ is related to the intrinsic Cauchy-Riemann operator $\bar\partial$ of J by*

$$(2.7.4) \qquad (\bar\partial^t_X Y,Z)=(\bar\partial_X Y,Z)+\frac{(t-1)}{4}((d^cF)^+_{X,Y,Z}-(d^cF)^+_{X,JY,JZ}).$$

In particular, $\bar\partial^t$ coincides with $\bar\partial$ for each t if (g,J) is $(2,1)$-symplectic, while, if (g,J) is not $(2,1)$-symplectic, $\bar\partial^t$ coincides with $\bar\partial$ if and only if $t=1$, i.e. $\nabla^t=\nabla^1$ is the Chern connection, or second canonical connection, of (g,J).

PROOF. – Easy consequence of (2.7.1), (2.7.4) and of the identity:

$$(2.7.5) \qquad (d^cF)^+_{X,Y,Z}-(d^cF)^+_{JX,Y,JZ}=((D_{JY}J)Z,X)-((D_YJ)Z,JX). \quad ■$$

REMARK 5. – Any J-linear connection, *a fortiori* any hermitian connection on M, induces a (unitary) connection on the *canonical bundle* $K_M^{-1}=\Lambda^m_{\mathbf{C}}TM$ of (M,J) (where TM is viewed as a complex vector bundle of rank m via J). This applies, in particular, to the canonical connections ∇^t. Let us still denote by ∇^t the induced connection on K_M^{-1}. From (2.5.5), we easily infer that ∇^t is

$$(2.7.6) \qquad \nabla^t s=\nabla^0 s+i\frac{t}{2}J\theta\otimes s,$$

for any section s of K_M^{-1} (where $i=\sqrt{-1}$).

3. – Dirac operators on an almost-hermitian manifold.

3.1. *Spin-structure on an almost-hermitian manifold.* The choice of a *Spin-structure* on an almost-complex manifold (M,g,J) is equivalent to the choice of a square-root $K_M^{1/2}$ of the canonical bundle K_M of the complex manifold (M,J).

We then have the following identification for the corresponding spinor bundle ΣM:

$$(3.1.1) \qquad \Sigma M=\Lambda^{0,*}M\otimes K_M^{1/2},$$

where: $\Lambda^{0,*}M=\sum_{p=0}^{m}\Lambda^{0,p}M$ denotes the bundle of *complex* forms of type $(0,*)$ (in (3.1.1), the tensor product is a $\mathbf{C}$-tensor product).

In particular, the spinor bundle ΣM splits as follows:

$$(3.1.2) \qquad \Sigma M=\bigoplus_{p=0}^{m}\Sigma^p M,$$

where, for each $p=0,\dots,m$, the sub-bundle $\Sigma^p M$ of *spinors of degree* p is identified with the bundle $\Lambda^{0,p}M\otimes K_M^{1/2}$ of $K_M^{1/2}$-valued $(0,p)$-forms. It is easily checked that $\Sigma^p M$ is also defined as the eigensub-bundle with respect to the eigenvalue $(2p-m)i$ for the Clifford action of the Kähler form F on ΣM.

A spinor field ξ of degree p is locally expressed as:

$$\xi=\psi\otimes\sigma, \tag{3.1.3}$$

where ψ is a form of type $(0,p)$ and σ a local, non-vanishing section of $K_M^{1/2}$. The identification of spinors of degree p with $K_M^{1/2}$-valued $(0,p)$-forms as well as the representation (3.1.3) will be systematically used in the sequel without further explanation.

The Clifford action of a (real or complex) vector field X on a spinor field ξ, denoted by a dot ., is given by:

$$X.\xi=\sqrt{2}((X^{1,0})^\flat\wedge\psi-X^{0,1}\lrcorner\psi)\otimes\sigma. \tag{3.1.4}$$

Here, $X=X^{1,0}+X^{0,1}$ is the usual type decomposition of X; $(X^{1,0})^\flat$ is the image of $X^{1,0}$ by the riemannian duality, extended by $\mathbb{C}$-linearity from the complexified tangent bundle $TM\otimes\mathbb{C}$ to the complexified cotangent spaces $T^*M\otimes\mathbb{C}$; $\lrcorner$ denotes the natural contraction of the complex vector field $X^{0,1}$ with the complex $(0,p)$-form ψ, defined by:

$$(X^{0,1}\lrcorner\psi)_{X_1,\dots,X_{p-1}}=\psi_{X^{0,1},X_1,\dots,X_{p-1}}, \tag{3.1.5}$$

for any (complex) vector fields $X_1,\dots,X_{p-1}$.

The corrective factor $\sqrt{2}$ ensures that the square of the action of X be equal to the multiplication by the scalar $-(X,X)$, where, here and henceforth, $(\cdot,\cdot)$ denotes the $\mathbb{C}$-bilinear extension of the inner product to the complexified tangent space.

The Clifford action of exterior forms on ΣM will also be denoted by a dot ..

WARNING. – The $\mathbb{C}$-linear riemannian duality from $TM\otimes\mathbb{C}$ to $T^*M\otimes\mathbb{C}$ *exchanges* types: $(X^{1,0})^\flat=(X^\flat)^{0,1}$. In particular, $(X^{1,0})^\flat$ is a $(0,1)$-form!

3.2. *Dirac operators.* The *riemannian Dirac operator*, denoted by P, acts on sections of ΣM as follows:

$$P\xi=\sum_{i=1}^n e_i.D_{e_i}\xi, \tag{3.2.1}$$

for any g-orthonormal frame $\{e_i\}_{i=1,\dots,n}$, where here D denotes the *induced* Levi-Civita connection on ΣM. In a similar way, for *any* hermitian connection ∇, the *hermitian Dirac operator* associated with ∇, also acting on ΣM, is defined by:

$$P^\nabla\xi=\sum_{i=1}^n e_i\cdot\nabla_{e_i}\xi, \tag{3.2.2}$$

where, again, ∇ denotes the induced connection on ΣM.

In contrast to the Levi-Civita connection, each hermitian connection ∇ is compatible with the identification (3.1.1) and the splitting (3.1.2), in the following sense:

$$\nabla_X\xi=\nabla_X\psi\otimes\sigma+\psi\otimes\nabla_X\sigma, \tag{3.2.3}$$

for any spinor field $\xi=\psi\otimes\sigma$ of degree p, where ∇ denotes the induced connections on ΣM in the l.h.s., on $\Lambda^{0,p}M$ and on $K_M^{1/2}$ in the r.h.s.

Then, the corresponding (hermitian) Dirac operator P^∇ splits as follows:

$$P^\nabla=P_+^\nabla+P_-^\nabla, \tag{3.2.4}$$

i.e., for any spinor field ξ of degree p, $P^\nabla\xi$ is the sum of a spinor field of degree $p+1$, denoted by $P_+^\nabla\xi$, and a spinor field of degree $p-1$, denoted by $P_-^\nabla\xi$. By (3.1.4), we have:

$$(P_+^\nabla\xi)_{Z_0,\dots,Z_p}=\sqrt{2}\sum_{j=0}^n(-1)^j(\nabla_{Z_j}\xi)_{Z_0,\dots,\hat{Z}_j,\dots,Z_p}, \tag{3.2.5}$$

$$(P_-^\nabla\xi)_{Z_1,\dots,Z_{p-1}}=-\sqrt{2}\sum_{i=1}^n(\nabla_{e_i}\xi)_{e_i,Z_1,\dots,Z_{p-1}}, \tag{3.2.6}$$

for any vector fields $Z_1,\dots,Z_p$ of type $(0,1)$ (as usual, the symbol $\hat{Z}_j$ indicates a missing argument).

3.3. *The Dolbeault operator.* The pre-holomorphic structure $\bar\partial$ defined by (3.7.1) on sections of TM, induces a pre-holomorphic structure, still denoted by $\bar\partial$, on any tensorial power of the canonical bundle K_M, in particular on $K_M^{1/2}$. By Proposition 5, we have:

$$\bar\partial\sigma = (\nabla^1)^{0,1}\sigma, \tag{3.3.1}$$

for any section σ of $K_M^{1/2}$, where, as usual, ∇^1 here denotes the connection on $K_M^{1/2}$ induced by the second canonical hermitian connection ∇^1.

We also denote by $\bar\partial$ the operator on sections of $\Lambda^{0,p}M$ defined by:

$$\bar\partial\psi = (d\psi)^{(0,p+1)}, \tag{3.3.2}$$

for any $(0,p)$-form ψ.

Upon combining these two operators, via (3.1.1) we obtain an operator $\bar\partial$ acting on sections of ΣM, defined by:

$$\bar\partial\xi = \bar\partial\psi\otimes\sigma + (-1)^p\psi\wedge\bar\partial\sigma, \tag{3.3.3}$$

for any spinor field $\xi = \psi\otimes\sigma$ of degree p.

We denote by $\bar\partial^*$ the *hermitian* adjoint of the operator $\bar\partial$ defined by (3.3.3) and by $\Box$ the (normalized) *Dolbeault operator* acting on sections of ΣM, defined by:

$$\Box = \sqrt{2}(\bar\partial + \bar\partial^*). \tag{3.3.4}$$

3.4. *Riemannian versus hermitian Dirac operators.* The riemannian Dirac operator P, the hermitian Dirac operators P^∇ and the Dolbeault operator $\Box$ have the same principal symbol, but don't coincide in general. The main goal of this and the following subsection is to compare these operators when ∇ is a canonical hermitian connection ∇^t, as defined in Definition 2. The corresponding (hermitian) connection P^{∇^t} will be simply denoted by P^t.

PROPOSITION 6. – *For any almost hermitian manifold (M,g,J) and any real t, the hermitian Dirac operator P^t, corresponding to the canonical hermitian connection ∇^t, is related to the riemannian Dirac operator D by:*

$$P^t\xi = P\xi - \frac{(t+1)}{4}\theta.\xi + \frac{(2t-1)}{4}(d^cF)^+.\xi + \frac{1}{4}(d^cF)^-.\xi. \tag{3.4.1}$$

In particular, P^t coincides with P for any t if and only if the almost-hermitian structure (g,J) is symplectic.

PROOF. – Easy consequence of (2.1.1), (2.1.5), (2.5.10) and (2.5.11). ■

3.5. *Riemannian Dirac operator versus Dolbeault operator.* In the Kähler case, it is well-known that the riemannian Dirac operator P coincides with the (normalized) Dolbeault operator $\Box$, cf. *e.g.* [7]. In the general case, we have:

PROPOSITION 7. – *For any almost-hermitian manifold (M,g,J), the riemannian Dirac operator P and the Dolbeault operator $\Box$ are related by:*

$$P\xi = \Box\xi + \frac{1}{4}(d^cF)^+.\xi - \frac{1}{4}(d^cF)^-.\xi. \tag{3.5.1}$$

In particular, P and $\Box$ coincide if and only if the almost-hermitian structure (g,J) is symplectic.

PROOF. – In order to compare P and $\Box$ it is convenient to first compare $\Box$ and the hermitian Dirac operators P^t, in particular, as we shall see, P^0 and P^1, by using (3.2.5)-(3.2.6). We first recall the expressions of the differential and the co-differential of a form with respect to any metrical connection with torsion.

LEMME 4. – *For any complex p-form ψ, any metrical connection ∇, of torsion T, any complex vector fields $X_0,\dots,X_p$. we have:*

$$(d\psi)_{X_0,\dots,X_p} = \sum_{j=0}^{p}(-1)^j(\nabla_{X_j}\psi)_{X_0,\dots,\hat X_j,\dots,X_p} + \sum_{j<k}(-1)^{j+k}\psi_{T_{X_j,X_k},X_0,\dots,\hat X_j,\dots,\hat X_k,\dots,X_p}, \tag{3.5.2}$$

$$(\delta\psi)_{X_1,\dots,X_{p-1}} = -\sum_{i=1}^{n}(\nabla_{e_i}\psi)_{e_i,X_1,\dots,X_{p-1}} + \psi_{(\mathrm{tr}\,T)^\natural,X_1,\dots,X_{p-1}} - \sum_{j=1}^{p-1}(-1)^j((X_j,T),\psi_{\cdot,\cdot,X_1,\dots,\hat X_j,\dots,X_{p-1}}), \tag{3.5.3}$$

where: $(\cdot,\cdot)$ denotes the $\mathbb{C}$-bilinear inner product induced by g on forms, with the convention precised by (1.1.1); $\mathrm{tr}\,T$ is the trace of the torsion T, i.e. the real 1-form defined by: $X \to \sum_{i=1}^{n}(e_i, T_{e_i,X})$; $(\mathrm{tr}\,T)^\natural$ is the dual vector field; (X_j,T) denotes the complex 2-form defined by: $Y,Z \to (X_j,T_{Y,Z})$; $\psi_{\cdot,\cdot,X_1,\dots,\hat X_j,\dots,X_{p-1}}$ denotes the complex 2-form de-

fined by: $Y, Z \to \psi_{Y,Z,X_1,\ldots,\hat{X}_j,\ldots,X_{p-1}}$; *the last line involves the inner product of these two 2-forms.*

PROOF. – Easy consequence of (2.1.1)-(2.1.5). Notice that (3.5.2) holds for *any* linear connection ∇ of torsion T. ■

We make use of Lemma 4 to evaluate $\bar{\partial}\psi$ and $\bar{\partial}^*\psi$ for ψ a $(0,p)$-form and ∇ a *hermitian* connection. In this case, the torsion T only occurs via its trace $\operatorname{tr} T$ and its $(2,0)$-part $T^{2,0}$, cf. II. To be precise, we have:

LEMME 5. – *For any $(0,p)$-form ψ, any hermitian connection ∇, any vector fields $Z_0, \ldots, Z_p$ of type $(0,1)$, we have:*

(3.5.4)
$$(\bar{\partial}\psi)_{Z_0,\ldots,Z_p} = \sum_{j=1}^{p} (-1)^j (\nabla_{Z_j}\psi)_{Z_0,\ldots,\hat{Z}_j,\ldots,Z_p} - \sum_{j<k} (-1)^{j+k} \psi_{T^{2,0}_{Z_j,Z_k}, Z_0,\ldots,\hat{Z}_j,\ldots,\hat{Z}_k,\ldots,Z_p},$$

(3.5.5)
$$(\bar{\partial}^*\psi)_{Z_1,\ldots,Z_{p-1}} = -\sum_{i=1}^{n} (\nabla_{e_i}\psi)_{e_i,Z_1,\ldots,Z_{p-1}} + \psi_{(\operatorname{tr}T)^\natural, Z_1,\ldots,Z_{p-1}} - \sum_{j=1}^{p-1} (-1)^j \big((Z_j, T^{2,0}), \psi_{\cdot,\cdot,Z_1,\ldots,\hat{Z}_j,\ldots,Z_{p-1}}\big).$$

In particular, if $\nabla = \nabla^0$ is the first canonical connection, the identities (3.5.4)-(3.5.5) are reduced to:

(3.5.6)
$$(\bar{\partial}\psi)_{Z_0,\ldots,Z_p} = \sum_{j=0}^{p} (-1)^j (\nabla^0_{Z_j}\psi)_{Z_0,\ldots,\hat{Z}_j,\ldots,Z_p},$$

(3.5.7)
$$(\bar{\partial}^*\psi)_{Z_1,\ldots,Z_{p-1}} = -\sum_{i=1}^{n} (\nabla^0_{e_i}\psi)_{e_i,Z_1,\ldots,Z_{p-1}} - \frac{1}{2}\psi_{\theta^\natural, Z_1,\ldots,Z_{p-1}}.$$

PROOF. – Direct consequence of Lemma 4, since the first canonical hermitian connection ∇^0 is characterized by $T^{2,0}$ being identically zero; on the other hand, the trace of the torsion of ∇^0 is equal to $-(1/2)\theta$ by (2.5.11). ■

LEMME 6. – *The Dolbeault operator $\square$ and the hermitian Dirac operator P^0 attached to the first canonical hermitian connection ∇^0 are linked together by:*

(3.5.8)
$$P^0\xi = \square\xi - \frac{1}{4}\theta.\xi.$$

PROOF. – Let $\xi = \psi \otimes \sigma$ be a section of ΣM of degree p. By (3.3.1), we have:

(3.5.9)
$$\square\xi = \square\psi \otimes \sigma + \sum_{i=1}^{n} (e_i.\psi) \otimes \nabla^1_{e_i}\sigma,$$

where, in the r.h.s., $\square$ denotes the usual (normalized) Dolbeault operator acting on scalar $(0,p)$-forms. From (3.5.9), (3.2.5)-(3.2.6), (3.5.6)-(3.5.7), we get:

(3.5.10)
$$\square\xi = \sum_{i=1}^{n} (e_i \cdot \nabla^0_{e_i}\psi \otimes \sigma + e_i.\psi \otimes \nabla^1_{e_i}\sigma) - \frac{\sqrt{2}}{2}(\theta^\natural \lrcorner \psi) \otimes \sigma.$$

By (2.7.6), for any section σ of $K_M^{1/2}$, we have:

(3.5.11)
$$\nabla^1\sigma = \nabla^0\sigma - \frac{1}{4} iJ\theta \otimes \sigma.$$

Then, (3.5.10) can be written as:

(3.5.12)
$$\square\xi = P^0\xi - \frac{1}{4} iJ\theta.\xi - \frac{\sqrt{2}}{2}(\theta^\natural \lrcorner \psi) \otimes \sigma,$$

which, by (3.1.4), is equivalent to (3.5.8). ■

Now, the identity (3.5.1) is a direct consequence of (3.5.8) and of the identity (3.4.1) for $t = 0$. This proves Proposition 6. ■

3.6. *Spin^c-Dirac operator versus Dolbeault operator.* Up to now, in this paragraph, M has been assumed to admit a *Spin*-structure, realized by a square-root $K_M^{1/2}$ of the canonical bundle K_M.

In general, *any* almost-complex manifold (M, J) admits a canonical *Spin^c*-structure, determined by the anti-canonical bundle $L = K_M^{-1}$ itself, cf. *e.g.* [1], [7]. The corresponding *Spin^c-spinor bundle*, denoted by $\Sigma^c M$, is then identified in a canonical way with the bundle $\Lambda^{0,*}M$ of $(0,*)$-forms, which, locally, should be viewed as the tensor product of the spinor bundle $\Sigma M = \Lambda^{0,*}M \otimes K_M^{1/2}$ by $L^{1/2} = K_M^{-1/2}$, which are only locally defined.

Via the metric g, the complex line bundle $L = K_M^{-1}$ can be regarded as a hermitian line bundle. Then, any unitary connection ∇ on L determines a *Spin^c-Dirac operator*, denoted by P^c, acting on sections of $\Sigma^c M$, locally identified with the riemannian Dirac operator P twisted by the induced connection on $L^{1/2}$.

In view of (3.5.9), it is convenient to express the auxiliary unitary connection ∇ on L as follows:

$$\nabla = \nabla^1 + ia\,, \qquad (3.6.1)$$

for some real 1-form a on M, where, again, ∇^1 denotes the unitary connection on L induced by the Chern connection ∇^1.

PROPOSITION 8. – *Let (M,g,J) be an almost-hermitian manifold of real dimension $n=2m>2$. Consider the canonical Spin^c-structure determined by the anti-canonical bundle $L=K_M^{-1}$, for which the Spin^c-spinor bundle is identified with the bundle $\Lambda^{0,*}M$ of $(0,*)$-forms. Let ∇ be any unitary connection on L expressed by* (3.6.1).

Then, the corresponding Spin^c-Dirac operator P^c is expressed by:

$$P^c\psi = \Box\psi + \frac{1}{4}(d^cF)^+\cdot\psi - \frac{1}{4}(d^cF)^-\cdot\psi + \frac{1}{2}ia\cdot\psi\,, \qquad (3.6.2)$$

where $\Box$ denotes the usual (normalized) Dolbeault operator $\sqrt{2}(\bar\partial + \bar\partial^*)$ acting on $(0,*)$-forms.

In particular, we have:

$$P^c\psi = \Box\psi + \frac{1}{2}ia\cdot\psi \qquad (3.6.3)$$

if and only if the almost-hermitian structure (g,J) is symplectic.

PROOF. – Since the *Spin^c*-Dirac operator P^c locally coincides with the riemannian Dirac operator P twisted by ∇, (3.6.2) is a direct consequence of (3.5.1), (3.5.8) and (3.5.11). The last statement is a direct consequence of (3.6.2). ∎

NOTE. – The fact that the *Spin^c*-Dirac operator P^c coincides, up to the contribution of the auxiliary unitary connection, with the normalized Dolbeault operator $\Box$ in the symplectic case, as in the Kähler case is here directly deduced from the general formula (3.6.2) and holds in any dimension. In dimension 4, it can also be obtained by indirect arguments and is used as a key fact in Taubes' treatment of four-dimensional symplectic manifolds in the framework of *Seiberg-Witten invariants*, cf. [9] for a general overview and references therein.

3.7. *The 4-dimensional case.* In the next corollary of Proposition

7, ε denotes the parity operator on spinors or forms, defined by:

$$\varepsilon(\psi) = \psi^+ - \psi^-\,, \qquad (3.7.1)$$

where ψ^+, resp. ψ^-, denotes the even, resp. odd, part of ψ with respect to the natural grading.

COROLLARY 3. – *Let (M,g,J) be an almost-hermitian manifold of real dimension 4. Then, the Spin^c-Dirac operator P^c with respect to the canonical Spin^c-structure and the auxiliary unitary connection $\nabla = \nabla^1 + ia$ is expressed by:*

$$P^c\psi = \Box\psi + \frac{1}{4}\theta.\varepsilon(\psi) + \frac{1}{2}ia.\psi\,, \qquad (3.7.2)$$

where θ is the Lee form of (g,J).

PROOF. – Since $n=4$, the component $(d^cF)^-$ vanishes identically for any almost-hermitian structure (g,J). Then, we have: $(d^cF)^+ = d^cF = J\theta\wedge F = J\theta.F-\theta$, so that (3.6.2) can be written as follows:

$$P^c\psi = \Box\psi + \frac{1}{4}J\theta.F.\psi - \frac{1}{4}\theta\cdot\psi + \frac{1}{2}ia.\psi\,. \qquad (3.7.3)$$

Recall that the Clifford action of the Kähler form F on $\Lambda^{0,p}M$ coincides with the multiplication with $(2p-2)i$.

If $\psi = (\psi)^-$ is of odd degree, *i.e.* is a section of $\Lambda^{0,1}M$, we have: $F\cdot\psi = 0$ and (3.7.3) is thus reduced to;

$$P^c\psi = \Box\psi - \frac{1}{4}\theta.\psi + \frac{1}{2}ia.\psi\,, \quad \forall\psi\in\Gamma(\Lambda^{0,1}M)\,. \qquad (3.7.4)$$

If $\psi = (\psi)^+$ is of even degree, we have: $\psi = \psi_0 + \psi_2$, where ψ_0, resp. ψ_2, is a section of $\Lambda^{0,0}M$, resp. $\Lambda^{0,2}M$. We have: $(J.\theta.F-\theta).\psi_0 = -(\theta + 2iJ\theta).\psi_0 = \theta.\psi_0$, because $F.\psi_0 = -2i\psi_0$ and the Clifford action of $J\theta$ on $\Lambda^{0,0}M$ coincides with the Clifford action of $i\theta$. In a similar way, we have: $(J\theta.F-\theta).\psi_2 = (2iJ\theta-\theta).\psi_2 = \theta.\psi_2$, because $F.\psi_2 = 2i\psi_2$ and the Clifford action of $J\theta$ on $\Lambda^{0,2}M$ coincides with the Clifford action of $-i\theta$. We thus get:

$$P^c\psi = \Box\psi + \frac{1}{4}\theta.\psi + \frac{1}{2}ia.\psi\,, \quad \forall\psi\in\Gamma(\Lambda^{0,0}M\oplus\Lambda^{0,2}M)\,. \qquad (3.7.5)$$ ∎

3.8. *Conformally invariant Dirac operators.* We begin this subsection by a few general considerations concerning «conformally invariant Dirac operators» on a general *conformal Spin*-manifold $(M,[g])$ of dimension n. In order to avoid the choice of a metric in the conformal class, the chosen *Spin*-structure is realized by a $CSpin(n)$-principal bundle $\tilde{Q}$ over M, where $CSpin(n) = \mathbb{R}^+ \times Spin(n)$ denotes the conformal spinorial group.

Then, for any real number k, the *spinor bundle of weight* k, denoted by $\Sigma^{(k)}M$, is defined as the complex vector bundle over M determined by $\tilde{Q}$ and the following representation of $CSpin(n)$ on the algebraic spinor space Σ: $(a,A).u = a^k A.u$, for any a in $\mathbb{R}^+$, any A in $Spin(n)$, u in Σ, where $A.u$ denotes the usual Clifford action of $Spin(n)$ on Σ.

We observe that the Clifford action of T^*M on weighted spinor bundles is still defined in the conformal framework, but sends each section of $\Sigma^{(k)}M$ into a section of $\Sigma^{(k-1)}M$.

Now, each conformal manifold (M,g) admits a distinguished family of linear connections, acting on sections of TM, namely the so-called *Weyl connections* defined as follows. A linear connection D on M is a Weyl connection (or a *Weyl structure* with respect to $c=[g]$), whenever D is torsion-free and preserves c: for any metric g in the conformal class, $Dg = -2\alpha^g \otimes g$ for some 1-form α^g. In this subsection, the Levi-Civita connection of g, a particular example of a Weyl connection with respect $[g]$, will be denoted by D^g. Then, D is related to D^g by:

$$D_X Y = D^g_X Y + \alpha^g_X Y + \alpha^g_Y X - g_{X,Y}(\alpha^g)^\natural \,, \tag{3.8.1}$$

where $(\alpha^g)^\natural$ is the dual vector field with respect to g; in a more abstract way, (3.8.1) can also be written as follows:

$$D_X = D^g_X + \alpha^g \otimes I + \alpha^g \wedge X \,, \tag{3.8.2}$$

where I denotes the identity of TM, and $\alpha^g \wedge X$ denotes the skew-symmetric endomorphism of TM defined by: $Y \to \alpha^g_Y X - g(X,Y)(\alpha^g)^\natural$; in particular, for any vector field X, $\alpha^g_X I$ is the *scalar part* and $\alpha^g \wedge X$ the *skew-symmetric part* of $D_X - D^g_X$. It follows that, for each weight k, the connection on the spinor bundle $\Sigma^{(k)}M$ induced by D is given by:

$$D_X \xi = D^g_X \xi + k\alpha^g_X \xi + \frac{1}{2}(\alpha^g \wedge X)\cdot\xi = \left(k - \frac{1}{2}\right)\alpha^g_X \xi - \frac{1}{2} X\cdot\alpha^g\cdot\xi \,, \tag{3.8.3}$$

which has to be understood as follows: for any choice of a metric g in the conformal class, each $\Sigma^{(k)}M$ is identified with the Riemannian spinor bundle ΣM and the r.h.s. of (3.8.3) is the *expression* of the Weyl connection D acting on sections of $\Sigma^{(k)}M$ with respect to this identification.Then the corresponding Dirac operator $P^{(D,k)}$, acting on sections of $\Sigma^{(k)}M$ with values in $\Sigma^{(k-1)}M$, obtained, as usual, by composing the Weyl connection D with the Clifford action of T^*M, is *expressed* as follows, with respect to any metric g in the conformal class:

$$P^{(D,k)}\xi = P\xi + \frac{(n-1+2k)}{2}\alpha.\xi \,. \tag{3.8.4}$$

In particular, for $k = -(n-1)/2$, we have:

$$P^{(D,(n-1)/2)} = P \,. \tag{3.8.5}$$

It follows that $P^{(D,(n-1)/2)}$ is independant of the chosen Weyl structure D, *i.e.* is canonically determined by the *conformal structure* c; in the sequel, the conformally invariant Dirac operator $P^{(D,(n-1)/2)}$ will be simply denoted by $P^{[g]}$. For any choice of a metric g in $[g]$, the expression of $P^{[g]}$ with respect to g coincides with the Riemannian Dirac operator $P^g = P$ of g. In other words, the Riemannian Dirac operator is *conformally covariant* with respect to the weight $-(n-1)/2$, cf. [7], meaning that for any g in $[g]$, the corresponding Riemannian Dirac operator P^g is the expression of a conformally well-defined operator $P^{[g]}$ from $\Sigma^{(-(n-1)/2)}M$ to $\Sigma^{(-(n+1)/2)}M$, via the identification of these two weighted spinor bundles with the Riemannian spinor bundle ΣM. This holds for any Spin-conformal manifold $(M,[g])$.

Let us consider the particular case that $(M,[g])$ is oriented, of even dimension $n=2m>2$, and admits a positive, $[g]$-orthogonal almost-complex structure J. For any metric g in $[g]$, we denote by θ^g the Lee form of the almost-hermitian structure (g,J). The main observation here, due to I. Vaisman, is that the Weyl connection D defined by:

$$\tilde{D}_X Y = D_X Y - \frac{1}{(n-2)}(\theta^g_X Y + \theta^g_Y X - (X,Y)(\theta^g)^\natural) \tag{3.8.6}$$

doesn't depend upon the choice og g in $[g]$. This connection is called the *canonical Weyl structure* or *canonical Weyl structure* of the conformal almost-hermitian structure $([g],J)$ and will be henceforth denoted by D (while, we recall, the Levi-Civita connection of some metric g is denoted by D^g).

The canonical Weyl connection D *doesn't* preserve J in general. More precisely [15]:

if $n > 4$, J is preserved by some Weyl connection if and only if J is integrable and the conformal hermitian structure $([g], J)$ is locally Kähler (equivalently, the Lee form θ is closed for any metric g in $[g]$); in that case, the canonical Weyl connection is the only Weyl connection preserving J;

if $n = 4$, J is preserved by some Weyl connection if and only if J is integrable; in that case, the canonical Weyl connection is the unique Weyl structure preserving J.

Since D is conformally invariant, for any weight k, the corresponding Dirac operator $P^{(D,k)}$, acting on sections of $\Sigma^{(k)}M$ with values in $\Sigma^{(k-1)}M$, is conformally invariant as well, and coincides with $P^{[g]}$ for $k = -(n-1)/2$.

In the same way as the riemannian Dirac operator has been said to be conformally covariant, cf. above, the hermitian Dirac operator P^t will be said *conformally invariant* for some weight κ, whenever, for any choice of a metric g in $[g]$, P^t coincides with some conformally invariant Dirac operator acting on sections of $\Sigma^{(\kappa)}M$ with values into $\Sigma^{(\kappa-1)}M$, via the identification of $\Sigma^{(\kappa)}M$ and $\Sigma^{(\kappa-1)}M$ with ΣM induced by g.

Then, we have, cf. also [10]

PROPOSITION 9. – (i) *The hermitian Dirac operator $P^{1/2}$, corresponding to the canonical hermitian connection $\nabla^{1/2}$, is conformally covariant with respect to the weight $\kappa = -(m+1)/4$.*

(ii) *If the Nijenhuis tensor of J satisfies the Bianchi identity (this condition is empty for $n = 4$), we have:*

$$P^{1/2} = P^{(D,-(m+1)/4)}, \tag{3.8.7}$$

i.e. for any metric g in $[g]$, the hermitian Dirac operator $P^{1/2}$ is the expression with respect to g of the conformally invariant Dirac operator $P^{(D,-(m+1)/4)}$ associated with the canonical Weyl connection D, acting on sections of $\Sigma^{-(m+1)/4}M$ with values into $\Sigma^{-(m+5)/4}M$.

PROOF. – Recall that the Nijenhuis tensor of J satisfies the Bianchi identity if and only if the 3-form $(d^cF)^-$ vanishes identically, cf. Proposition 1.

From (3.4.1), for any metric g in $[g]$, we have:

$$P^{1/2}\xi = P^g\xi - \frac{3}{8}\theta^g.\xi + \frac{1}{4}(d^cF)^-.\xi, \tag{3.8.8}$$

for any section ξ of the riemannian spinor bundle ΣM.

For any other metric $g' = \phi^{-2}g$ in $[g]$, we have: $(d^cF')^- = \phi^{-2}(d^cF)^-$, so that $(1/4)(d^cF)^-\cdot\xi$ is the expression with respect to g of a conformally invariant zero-order operator from $\Sigma^{(k)}M$ into $\Sigma^{(k-1)}M$ for any k. On the other hand, by (3.8.4) $P^g\xi - (3/8)\theta^g.\xi$ is the expression with respect to g of $P^{(D,\kappa)}$ for the value of κ determined by: $3/8 = (n-1+2\kappa)/(2(n-2))$, *i.e.* for $\kappa = -(m+1)/4$. ■

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