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A POSITIVITY PROPERTY FOR FOLIATIONS ON COMPACT KÄHLER MANIFOLDS

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We prove that the canonical bundle of a foliation by curves on a compact Kähler manifold is pseudoeffective, unless the foliation is a (special) foliation by rational curves.

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1. Introduction

Let $\mathcal{F}$ be a holomorphic (possibly singular) foliation by curves on a compact connected Kähler manifold X. Following the uniformization results of [4], we may distinguish three possible situations:

(i) $\mathcal{F}$ is a *hyperbolic foliation*: most leaves, i.e. all the leaves outside a pluripolar subset, are uniformized by the disc $\mathbb{D}$.
(ii) $\mathcal{F}$ is a *strictly parabolic foliation*: all the leaves are uniformized by the affine line $\mathbb{C}$.
(iii) $\mathcal{F}$ is a *rational quasi-fibration*: all the leaves are uniformized by the projective line $\mathbb{P}$.

This trichotomy is related to positivity properties of the canonical bundle $K_{\mathcal{F}}$ of the foliation $\mathcal{F}$. For a hyperbolic foliation, its canonical bundle is *pseudoeffective*, in the sense that it admits a (singular) hermitian metric whose curvature is a closed positive current: this is the main result of [4], asserting that such a metric is provided by the leafwise Poincaré metric. On the opposite side, for a rational quasi-fibration the canonical bundle is certainly *not* pseudoeffective: a generic leaf is a smooth rational curve free of singularities, over which the canonical bundle has degree -2, and this prevents pseudoeffectivity. We refer to [3] and references therein for the basic notions related to pseudoeffective line bundles.

In this paper, we will be concerned with the intermediate case, the case of strictly parabolic foliations. Here it is easy to produce examples in which the canonical bundle is pseudoeffective, or even ample, see, for instance, [5]. But also there are examples in which this pseudoeffectivity fails, for instance, the radial foliation on the projective space.

Recall that a foliation $\mathcal{F}$ on X is said to be a *foliation by rational curves* if for every $x \in X$ there exists a rational curve through x and tangent to $\mathcal{F}$. Because these rational curves may pass through $\mathrm{Sing}(\mathcal{F})$, such a class of foliations is larger than the class of rational quasi-fibrations, see [4, p. 146]. It can even happen that a foliation by rational curves is of hyperbolic type. Anyway, positivity properties of the canonical bundle of a foliation by rational curves are quite easy to establish, and moreover such a foliation can be reduced, by blowing-up, to a rational quasi-fibration, for which pseudoeffectivity of the canonical bundle is definitely lost. Therefore, we shall concentrate on the complementary class.

Theorem 1.1. *Let $\mathcal{F}$ be a holomorphic foliation by curves on a compact connected Kähler manifold X. Suppose that $\mathcal{F}$ is not a foliation by rational curves. Then its canonical bundle $K_{\mathcal{F}}$ is pseudoeffective.*

When X is projective, this result is not new. Basically, it is a theorem by Miyaoka [9], as reworked by Shepherd–Barron [11]. Their proof is based on positive characteristic techniques. A purely characteristic zero proof, but still demanding the projectivity of X, has been given by Bogomolov and McQuillan [2]. Strictly speaking, in these papers the result is slightly weaker, because there the conclusion is only that $K_{\mathcal{F}}$ is a so called "almost nef" line bundle, a property possibly weaker than pseudoeffectivity. However, it is a result of [3] that, on projective manifolds, almost nefness and pseudoeffectivity are in fact the same thing.

Our proof is in part inspired by [2], but since the beginning we need to replace almost nefness (which is almost meaningless on nonprojective manifolds) by pseudoeffectivity. In this way, even in the projective case we hope that our proof is more natural and direct than the already existing ones, because we avoid the deep contribution of [3]. Besides results from [4, 5], we shall need an extension theorem for meromorphic maps into Kähler manifolds which is largely based on a work by Dingoyan [7]. In some sense, this extension theorem replaces Andreotti's theorem used (implicitely) in [2].

As already observed, when $\mathcal{F}$ is hyperbolic then the theorem above is contained in [4], in an effective form: we have an explicit metric on $K_{\mathcal{F}}$ (the leafwise Poincaré metric) giving the pseudoeffectivity. Unfortunately, such an effectiveness is lost when $\mathcal{F}$ is strictly parabolic: our result above is only existential. One could naively hope for a plurisubharmonic metric on $K_{\mathcal{F}}$ which is flat and complete along the leaves, but examples of strictly parabolic foliations with ample canonical bundle destroy this sought. We refer to [5] for some partial results in this direction.

We shall not pursue this theme here, but let us notice that simple variations on the theorem above give also some informations on the discrepancy between pseudoeffectivity and nefness of $K_{\mathcal{F}}$. Indeed, suppose that $K_{\mathcal{F}}$ is pseudoeffective but not nef. Then [3] there exists a maximal countable collection of proper irreducible analytic subsets $Y_j \subset X$ such that the restriction $K_{\mathcal{F}}|_{Y_j}$ is not pseudoeffective, for every j. If we further assume that Y_j is not entirely contained in the singular set of the foliation, then we obtain that all the leaves of $\mathcal{F}$ through points of Y_j compactify to rational curves (probably these leaves are contained in Y_j, i.e. each Y_j is $\mathcal{F}$-invariant, but we did not found a formal proof of this fact).

Let us finally observe the following application of our theorem to the classification of Kähler threefolds.

Corollary 1.2. *Let X be a compact connected nonprojective Kähler threefold. Suppose that its canonical bundle K_X is not pseudoeffective. Then X is covered by rational curves.*

The proof is the same as for [4, Corollary 1.2], exploiting the fact that (by Kodaira nonembedding theorem) on any compact connected nonprojective Kähler threefold we have by free a foliation $\mathcal{F}$ with

$$K_X = K_{\mathcal{F}} \otimes \mathcal{O}_X(D), \quad D \geq 0.$$

The result is in fact true also for projective threefolds, basically by Miyaoka and Mori [10, 3]. Remark that, in the nonprojective case, if X is covered by rational curves then up to a bimeromorphism X is a $\mathbb{P}$-bundle over a surface (Kähler and nonprojective). On the other hand, when K_X is pseudoeffective then, according to the abundance conjecture [3], one expects that the Kodaira dimension of X is nonnegative, and this should complete the canonical classification of compact Kähler nonprojective threefods.

2. Compactification of Meromorphic Maps from Line Bundles

In this section, we shall prove an extension theorem for meromorphic maps. The proof will follow arguments extracted from [7], plus a suitable interpretation of the (non-)pseudoeffectivity property of a line bundle.

Let us firstly recall the Monge–Ampère estimate from [7] that we shall need, in a simplified form.

Let V be a connected complex manifold, of dimension n, and let ω be a closed positive $(1,1)$-current on V which locally admits continuous potentials (for instance, ω may be any smooth closed semipositive $(1,1)$-form). Let U be an open relatively compact subset of V. Define

$$P_\omega(V) = \{\phi : V \to [-\infty, +\infty) \text{ } u.s.c. \mid dd^c\phi + \omega \geq 0\}$$

and

$$P_\omega(V, U) = \{\phi \in P_\omega(V) \mid \phi|_U \leq 0\}.$$

Consider the subset $W \subset V$ where the family of functions $P_\omega(V, U)$ is locally bounded from above. By definition, it is an open subset which contains U.

Hypothesis 2.1. $W = V$.

Under this assumption, we may introduce the real function on V

$$\phi^* = \text{upper regularization of} \sup_{\phi \in P_\omega(V,U)} \phi$$

which obviously is everywhere nonnegative, and identically zero on U.

Hypothesis 2.2. $\phi^* : V \to \mathbb{R}^+$ is exhaustive, i.e. $\{\phi^* < \lambda\}$ is relatively compact in V for every $\lambda \in \mathbb{R}^+$.

Under these two assumptions, we have [7, Main theorem]:

$$\int_V \omega^n < +\infty.$$

For sake of completeness and reader's convenience, let us recall the scheme of the proof. The above extremal function ϕ^* still belongs to $P_\omega(V,U)$, and moreover outside the closure of U it satifies the Monge-Ampère equation $(dd^c\phi^* + \omega)^n \equiv 0$, basically by [1] ("balayage"). Because the closure of U is compact, we therefore have

$$\int_V (dd^c\phi^* + \omega)^n < +\infty.$$

On the other hand, using the fact that ϕ^* is exhaustive, a Stokes type argument shows [7, Lemma 4]

$$\int_V \omega^n \leq \int_V (dd^c\phi^* + \omega)^n,$$

whence the finiteness of the first integral.

In our context, in order to check Hypothesis 2.1, we shall need the following lemma on pseudoeffective line bundles. Let X be a compact connected complex manifold and let L be a line bundle over X. We shall denote by E the total space of L, and by $\Sigma \subset E$ the graph of its zero section. Recall that a line bundle is *pseudoeffective* if it admits a (singular) hermitian metric whose curvature is a closed positive current, that is, whose local weights are plurisubharmonic functions. Recall also that an open subset W of a complex manifold V is *locally pseudoconvex* if there exists a Stein open covering of V whose trace on W is still Stein.

Lemma 2.3. *Suppose that there exists a neighbourhood W of Σ in E, $W \neq E$, which is locally pseudoconvex in E. Then the dual line bundle L^* is pseudoeffective.*

Proof. This is an easy application of Yamaguchi's variation formula [13] (see also [8]).

Take a point $x \in X$ and a vector $\xi \in L_x^*$. We may identify ξ with a cotangent vector to $W_x = W \cap E_x$ at the point $0_x = \Sigma \cap E_x$. Then we may set

$\|\xi\|$ = the Poincaré norm of ξ in the Riemann surface W_x.

Remark that $\|\xi\| = +\infty$ if $W_x = E_x \simeq \mathbb{C}$ or $W_x = E_x \backslash \{1 \text{ point}\} \simeq \mathbb{C}^*$, and $\|\xi\| \in (0, +\infty)$ otherwise.

If $B \subset X$ is a small Stein neighborhood of x then $\cup_{y \in B} W_y$ is Stein, by its local pseudoconvexity inside $\cup_{y \in B} E_y \simeq B \times \mathbb{C}$. Therefore, by [13, 8], the metric $\| \cdot \|$ so defined on L^* has positive curvature, in the sense of currents, provided it is not identically $+\infty$.

If $\| \cdot \| \equiv +\infty$ then, for every $x \in X$, either $W_x = E_x$ or $W_x = E_x \setminus \{p_x\}$. Using the local pseudoconvexity of W, we see that the second possibility occurs over some open subset $\Omega \subset X$, in such a way that the map $\Omega \ni x \mapsto p_x$ is holomorphic and its graph is closed in E. By assumption, Ω is not empty. By Radó theorem [6], $X \setminus \Omega$ is an hypersurface in X and the above map extends meromorphically to the full X, with that hypersurface as set of poles. Thus we obtain a meromorphic section of L, without zeroes because $W \supset \Sigma$, and hence a holomorphic section of L^*. Therefore L^* is (pseudo)effective. □

Note that the converse to this lemma is also true (and easy): L^* pseudoeffective implies the existence of a nonfull locally pseudoconvex (disqued) neighbourhood of Σ, given by the open unit ball of the dual metric. In fact, the lemma above admits a simpler proof: setting W_θ = rotation of W by the angle θ and W' = the connected component of $\cap_\theta W_\theta$ which contains Σ, then W' is still a nonfull locally pseudoconvex neighborhood of Σ, and moreover it is disqued; hence, by considering it as the open unit ball of a metric on L, we obtain by duality a metric on L^* with positive curvature (recall that a subset of $B \times \mathbb{C}$ of the type $\{(z, w) \mid |w| \leq e^{-f(z)}\}$ is open and locally pseudoconvex if and only if f is plurisubharmonic). However, we think that the proof above is less tricky and more apted to eventual generalizations, as evoked in the remark at the end of this section.

Consider now the following data:

- X a compact connected Kähler manifold;
- L a line bundle over X, with total space E and zero section $\Sigma \subset E$;
- $F = E \cup H$ the compact manifold, ruled over X, obtained by glueing to E the section at infinity H;
- $f : E \dashrightarrow M$ a meromorphic map into some compact Kähler manifold M.

It may be worth noting that the normal bundle of H in F is isomorphic to L^*.

Proposition 2.4. *Suppose that L^* is not pseudoeffective. Then f extends through H to a meromorphic map $f' : F \dashrightarrow M$.*

Proof. Fix a Kähler form ω_2 on M. The manifold F also admits a Kähler form, denoted by ω_1. On the product $F \times M$ we put the Kähler form $\omega = \pi_F^*(\omega_1) + \pi_M^*(\omega_2)$, where π_F and π_M are the projections to F and M.

Let

$$\Gamma_f \subset F \times M$$

be the graph of $f : E \dashrightarrow M$, which is an analytic subset of $E \times M$. Our aim is to prove that its volume

$$\mathrm{Vol}(\Gamma_f) = \int_{\Gamma_f} \omega^n$$

(where n is the dimension of E) is finite, so that by Bishop's theorem [6], its closure in $F \times M$ is still an analytic subset, of the same dimension. This gives the required extension of the map f [12].

The graph Γ_f could be singular, so, to be safe, let us replace it by a resolution

$$\pi : \tilde{\Gamma}_f \to \Gamma_f.$$

We shall work with the projection

$$\tilde{\pi}_F = \pi_F \circ \pi : \tilde{\Gamma}_f \to E,$$

which realizes an analytic isomorphism between $\tilde{\Gamma}_f \backslash Z$ and $E \backslash B$, for suitable analytic subsets $Z \subset \tilde{\Gamma}_f$ and $B \subset E$. Set

$$\tilde{\omega} = \pi^*(\omega|_{\Gamma_f}),$$

a smooth closed semipositive $(1,1)$-form on $\tilde{\Gamma}_f$. Note that its direct image $\bar{\omega} = (\tilde{\pi}_F)_*(\tilde{\omega})$ $(= \omega_1 + f^*(\omega_2))$ is a closed positive current on E, smooth on $E \backslash B$.

Take a relatively compact neighborhood U of Σ in E, and set $\tilde{U} = \tilde{\pi}_F^{-1}(U)$. Consider the space

$$P_{\tilde{\omega}}(\tilde{\Gamma}_f, \tilde{U})$$

as defined at the beginning of this section. Let $\hat{U} \subset \tilde{\Gamma}_f$ be the open subset where this family of functions is locally bounded from above.

Claim 2.5. $\hat{U} = \tilde{\Gamma}_f$.

Indeed, consider the isomorphism $j : \tilde{\Gamma}_f \backslash Z \to E \backslash B$ induced by restricting $\tilde{\pi}_F$. Let $\mathcal{P}$ be the set of functions on $E \backslash B$ of the form $\phi \circ j^{-1}$, $\phi \in P_{\tilde{\omega}}(\tilde{\Gamma}_f, \tilde{U})$. Then, obviously, $j(\hat{U} \backslash Z)$ is equal to the open subset of $E \backslash B$ where the family $\mathcal{P}$ is locally bounded from above. Clearly $\mathcal{P} \subset P_{\bar{\omega}}(E \backslash B)$, and so, by [7, Lemmata 6–7], $j(\hat{U} \backslash Z)$ is locally pseudoconvex in $E \backslash B$ and the interior of $j(\hat{U} \backslash Z) \cup B$ is locally pseudoconvex in E. This interior contains U and hence Σ, therefore by Lemma 2.3 it must coincide with the full E. It follows that $\hat{U}$ contains the full $\tilde{\Gamma}_f \backslash Z$. From this last fact we deduce that $\hat{U} = \tilde{\Gamma}_f$, because a family of psh functions which is locally bounded from above outside an analytic subset is locally bounded from above also on the same analytic subset, by the submean inequality (our functions are not psh, but locally they are of the type (psh)-(a fixed local potential of $\tilde{\omega}$), so the same criterion applies).

Consider now the extremal function $\phi^* : \tilde{\Gamma}_f \to \mathbb{R}^+$, upper regularization of $\sup_{P_{\tilde{\omega}}(\tilde{\Gamma}_f, \tilde{U})} \phi$.

Claim 2.6. *ϕ^* is exhaustive.*

Indeed (see also [7, p. 233]), take on E (as a subset of F) the function $\psi_0 = -\log \mathrm{dist}_H$, where dist_H is the distance from H with respect to the Kähler metric ω_1. Because H is an hypersurface in F, standard estimates give a positive constant C such that $dd^c\psi_0 \geq -C\omega_1$. Thus, the function $\psi = \frac{1}{C}\psi_0 \circ \tilde{\pi}_F$ on $\tilde{\Gamma}_f$ belongs to $P_{\tilde{\omega}}(\tilde{\Gamma}_f)$:

$$dd^c\psi \geq -\tilde{\pi}_F^*(\omega_1) \geq -\tilde{\omega}.$$

By adding a (negative) constant, we may also achieve $\psi \leq 0$ on $\tilde{U}$, that is $\psi \in P_{\tilde{\omega}}(\tilde{\Gamma}_f, \tilde{U})$. By definition, ϕ^* is not smaller than ψ, and the exhaustivity of the former follows from the one of the latter.

By Claims 2.5 and 2.6, we may now apply Dingoyan's estimate:

$$\int_{\tilde{\Gamma}_f} \tilde{\omega}^n < +\infty.$$

Hence $\mathrm{Vol}(\Gamma_f)$ is finite, and the proof is complete. □

Remark 2.7. It is not difficult to obtain a local version of this result, in which the map f is initially defined only on a neighborhood of H minus H, instead of the full E. Also, it is conceivable a "nonlinear" generalization, for a meromorphic map $f : Y\backslash H \dashrightarrow M$ where Y is any Kähler manifold and H is any compact connected hypersurface with nonpseudoeffective normal bundle. When H is projective, this can be proved with the help of [3], which guarantees the existence of a covering family of curves in H over which the normal bundle of H has strictly negative degree (so that we are reduced to extending maps defined outside a normal surface singularity).

3. Compactification of Parabolic Foliations

Let X be a compact connected Kähler manifold and $\mathcal{F}$ a holomorphic foliation by curves on X. Set $X^0 = X\backslash\mathrm{Sing}(\mathcal{F})$. According to [4, Lemma 2.1], there exists a complex manifold $U_{\mathcal{F}}$, called *covering tube*, with the following properties:

(i) there exists a holomorphic submersion

$$P : U_{\mathcal{F}} \to X^0$$

and a holomorphic section

$$p : X^0 \to U_{\mathcal{F}}$$

such that, for every $x \in X^0$, $(P^{-1}(x), p(x))$ is the universal covering of the leaf L_x through x with basepoint x;

(ii) there exists a meromorphic map

$$\pi : U_{\mathcal{F}} \dashrightarrow X$$

sending $(P^{-1}(x), p(x))$ onto (L_x, x).

In fact, this is done in [4] in a local terminology, using a local transversal T to $\mathcal{F}$ instead of the full X^0. The formulation above can be immediately obtained either by repeating the arguments, mutatis mutandis, or by a patching procedure. Remark that the local product structure of $\mathcal{F}$ over X^0 induces a partial local product structure of $U_{\mathcal{F}}$: if $T \subset X^0$ is a local transversal to the foliation and $B \simeq T \times \mathbb{D} \subset X^0$ is an open subset over which $\mathcal{F}$ is a product, then the restriction of $U_{\mathcal{F}}$ over B is isomorphic to $U_T \times \mathbb{D}$, where U_T is the restriction of $U_{\mathcal{F}}$ over $T \times \{0\} \subset B$ (i.e. U_T is the covering tube of [4]). This is because fibres of $U_{\mathcal{F}}$ over points in the same leaf differ only by their basepoints.

The tube $U_{\mathcal{F}}$ is a sort of "integrated" tangent bundle to $\mathcal{F}$ (over X^0) with zero section given by p. The meromorphic map $\pi : U_{\mathcal{F}} \dashrightarrow X$ is the integrated form of the morphism $T_{\mathcal{F}} \to TX$, which is the differential equation defining the foliation. In fact, this last morphism is the linearization of π along the hypersurface $p(X^0) \subset U_{\mathcal{F}}$ (by construction, π is holomorphic around such a hypersurface).

Suppose now that $\mathcal{F}$ is a *strictly parabolic* foliation in the sense of [5]: the universal covering of every leaf is isomorphic to $\mathbb{C}$. Denote by $E_{\mathcal{F}}$ the total space of the tangent bundle $T_{\mathcal{F}}$, and by $\Sigma_{\mathcal{F}} \subset E_{\mathcal{F}}$ its zero section.

Proposition 3.1. *If $\mathcal{F}$ is strictly parabolic, then there exists a* (*natural*) *biholomorphism*

$$\varphi : E_{\mathcal{F}}|_{X^0} \to U_{\mathcal{F}}$$

sending fibres to fibres and $\Sigma_{\mathcal{F}}|_{X^0}$ to $p(X^0)$.

Proof. Take $x \in X^0$ and let E_x be the fibre of $E_{\mathcal{F}}$ over x. By definition, this is the tangent space to L_x at the point x, and it can also be identified with the tangent space to $P^{-1}(x)$ at the point $p(x)$. Because $P^{-1}(x)$ is isomorphic to $\mathbb{C}$, there exists a *unique* isomorphism

$$\varphi_x : E_x \to P^{-1}(x)$$

with $\varphi_x(0) = p(x)$ and $\varphi_x'(0) = \mathrm{id}$, via the identifications above. By glueing together these various φ_x, $x \in X^0$, we obtain the desired map $\varphi : E_{\mathcal{F}}|_{X^0} \to U_{\mathcal{F}}$.

Of course, and this is the main point, we need to check that the map φ so defined is holomorphic, that is, that the affine structure on the leaves of $\mathcal{F}$ (used to construct φ) varies in a holomorphic way with respect to the leaf. But this is just [5, Theorem 1], which affirms that over a (small) polydisc $B \subset X^0$ the covering tube $U_{\mathcal{F}}$ has a product structure:

$$U_{\mathcal{F}}|_B \simeq B \times \mathbb{C}. \qquad \square$$

By composing φ with π we obtain a meromorphic map from $E_{\mathcal{F}}|_{X^0}$ into X, sending fibres into leaves. The analytic subset $E_{\mathcal{F}} \backslash E_{\mathcal{F}}|_{X^0}$ $(= E_{\mathcal{F}}|_{\mathrm{Sing}(\mathcal{F})})$ has codimension at least 2 in $E_{\mathcal{F}}$, and so a standard extension result [12] allows to extend this map to the fibres over $\mathrm{Sing}(\mathcal{F})$. Hence, we finally obtain a meromorphic map

$$\Phi : E_{\mathcal{F}} \dashrightarrow X$$

sending fibres to leaves. This map can be thought as a "skew flow" generating the foliation, defined on $E_{\mathcal{F}}$, the skew product of X and the "time" $\mathbb{C}$.

We are now ready to prove the theorem stated in the introduction. As explained there, it is sufficient to consider only the case of strictly parabolic foliations.

Theorem 3.2. *Let X be a compact connected Kähler manifold and let $\mathcal{F}$ be a strictly parabolic foliation on X. Suppose that the canonical bundle is not pseudo-effective. Then $\mathcal{F}$ is a foliation by rational curves.*

Proof. Consider the map $\Phi : E_{\mathcal{F}} \dashrightarrow X$ constructed above. By Proposition 2.4, if $K_{\mathcal{F}} = T^*_{\mathcal{F}}$ is not pseudoeffective then Φ can be meromorphically extended to the section at infinity of $E_{\mathcal{F}}$. Because Φ sends fibres to leaves, the conclusion follows. □

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