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Subharmonic variation of the leafwise Poincaré metric

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1. Introduction

Let X be a compact complex algebraic surface and let $\mathcal{F}$ be a holomorphic foliation, possibly with singularities, on X. On each leaf of $\mathcal{F}$ we put its Poincaré metric (this will be defined below in more precise terms). We thus obtain a (singular) hermitian metric on the tangent bundle $T_{\mathcal{F}}$ of $\mathcal{F}$, and dually a (singular) hermitian metric on the canonical bundle $K_{\mathcal{F}} = T^*_{\mathcal{F}}$ of $\mathcal{F}$. The main aim of this paper is to prove that this metric on $K_{\mathcal{F}}$ has *positive* curvature, in the sense of currents. Of course, the positivity of the curvature in the leaf direction is an immediate consequence of the definitions; the nontrivial fact is that the curvature is positive also in the directions *transverse* to the leaf. This last fact can be rephrased by saying that the Poincaré metric on the leaves has a *subharmonic variation*.

In order to give more precise statements, let us firstly set out the types of singularities that will be allowed. Concerning X, we shall not require that X be smooth, but (for a reason which will be clear later) we will allow *Hirzebruch–Jung singularities*: around such a singularity, X looks like a quotient of the ball $\mathbf{B}^2 \subset \mathbf{C}^2$ by a linear action of a finite cyclic group [BPV, pp. 80–84]. These are very mild singularities: they are rational and X is even projective [Art]; but we shall not need these facts. Concerning $\mathcal{F}$, we shall require only that its singularities $Sing(\mathcal{F})$ are isolated and disjoint from $Sing(X)$, in the sense that around a point in $Sing(X)$ the foliation looks like a quotient of a regular foliation on $\mathbf{B}^2$. We shall set

$$X' = X \setminus Sing(\mathcal{F}) \qquad\qquad X'' = X' \setminus Sing(X)$$

and, unless otherwise stated, it will always be supposed that $Sing(X)$ and $Sing(\mathcal{F})$ satisfy the previous assumptions.

Let us also precise the notion of **leaf** of a foliation. As usual in foliation theory, it is sufficient to define local leaves and then to patch local leaves to obtain global ones. If $p \in X''$ then the definition of local leaf through p is the usual one. If $p \in Sing(X)$ then, locally, X is like $\mathbf{B}^2/\mathbf{Z}_n$, where the $\mathbf{Z}_n$-action on $\mathbf{B}^2$ is generated by $(z, w) \mapsto (e^{2\pi i \frac{q}{n}} z, e^{2\pi i \frac{1}{n}} w)$ for some $q \in \{1, ..., n-1\}$ prime to n, and $\mathcal{F}$ is the quotient of the vertical foliation $\{dz = 0\}$ on $\mathbf{B}^2$. It is therefore natural to define the local leaf of $\mathcal{F}$ through p as the projection of $\{z = 0\}$ under $\mathbf{B}^2 \to \mathbf{B}^2/\mathbf{Z}_n$. Hence this local leaf is a disc centered at p, but it must be considered as an *orbifold* with p affected by the multiplicity n. Remark that the holonomy of $\mathcal{F}$ along a cycle in this local leaf which turns one time around p is *not* the identity, but it is periodic of period exactly equal to n. Finally, if $p \in Sing(\mathcal{F})$ then the local leaf through p is, by definition, the single point $\{p\}$.

For every $p \in X$ we therefore have a well defined leaf of $\mathcal{F}$ through p, which is $\{p\}$ if $p \in Sing(\mathcal{F})$ and a connected orbifold L_p containing p and immersed in X' if $p \in X'$. It is easy to see that L_p is never a so-called "bad orbifold", i.e. L_p always admits a universal covering $\tilde{L}_p$, in the sense of orbifolds (indeed, the holonomy covering $\hat{L}_p$ is well defined and has already all its points of multiplicity 1). Therefore we may put on each L_p, $p \in X'$, its **Poincaré metric**: the unique complete metric of curvature -1 if $\tilde{L}_p$ is the unit disc $\mathbf{D}$, and the identically vanishing metric if $\tilde{L}_p$ is $\mathbf{C}$ or $\mathbf{C}P^1$. Alternatively, this metric can be defined *à la* Kobayashi [Kob], by considering holomorphic maps $\mathbf{D} \to L_p$ with appropriate ramifications over points in $L_p \cap Sing(X)$.

Let us look at this metric in a local chart. Take $p \in X''$ and fix local coordinates (z, w) centered at p such that $\mathcal{F} = \{dz = 0\}$. The leafwise Poincaré metric (or, more precisely, its area form) is then locally expressed as

$$e^{F(z,w)} i dw \wedge d\bar{w}$$

for some function F with values in $[-\infty, +\infty)$. Restricted to a leaf $\{z = z_0\}$ the function F is either identically $-\infty$ (parabolic leaf) or a real analytic function which satisfies the differential equation

$$\frac{\partial^2 F}{\partial w \partial \bar{w}} = e^F$$

expressing "curvature $= -1$" (hyperbolic leaf). Remark that $F(z_0, \cdot)$ is subharmonic (or identically $-\infty$) for every fixed z_0. A priori, we have no information concerning the transverse regularity of F. It is however easy to see, using the Kobayashi–type definition, that F is upper semicontinuous (see also [Can] and [Suz]). Our theorem will say that F is plurisubharmonic (unless identically $-\infty$), provided that a natural hypothesis which we now explain is satisfied.

Consider the **canonical bundle** $K_{\mathcal{F}}$ of $\mathcal{F}$ [McQ] [Bru]. Strictly speaking, this is not a line bundle (unless X is smooth) because it is not locally free around a point in $Sing(X)$: in the notation above, the vertical foliation $\{dz = 0\}$ on $\mathbf{B}^2$ is $\mathbf{Z}_n$-invariant, but the vector field $\frac{\partial}{\partial w}$ generating that foliation is not, as well as any other (nonvanishing) vector field generating that foliation. However, $K_{\mathcal{F}}$ is always a **Q**-bundle (i.e. an element of $Pic(X) \otimes \mathbf{Q}$), and hence it is as good as a genuine line bundle. For instance, there is no problem in speaking of "hermitian metric" on $K_{\mathcal{F}}$. The basic assumption that we will need for our theorem is that $K_{\mathcal{F}}$ is **nef**, that is it has nonnegative degree $K_{\mathcal{F}} \cdot C$ on every algebraic curve $C \subset X$. We shall say, in that case, that $\mathcal{F}$ is a **nef foliation**. Note that if $\mathcal{F}$ is a nef foliation then every leaf L has universal covering isomorphic to **D** or to **C**: from $\tilde{L} \simeq \mathbf{C}P^1$ it easily follows that L is an algebraic curve over which $K_{\mathcal{F}}$ has negative degree ($= -\chi(L)$). Fundamental results of Miyaoka [Miy] and McQuillan [McQ] (see also [Bru]) lead to the following: *every* foliation on a compact complex algebraic surface is birational either to a nef foliation or to a (trivial) fibration with rational fibres. Therefore, it is not a serious loss of generality to restrict our attention to nef foliations. Remark that, even if we start with a foliation on a *smooth* surface, the corresponding nef models will be defined, generally speaking, on *singular* surfaces (with Hirzebruch–Jung singularities). In many cases one can also set out a unique "minimal" nef model, but we shall not need to work with such a special representative.

It is convenient to look at the leafwise Poincaré metric as a singular hermitian metric on $K_{\mathcal{F}}$. Around $p \in X''$ we may choose a local nonvanishing holomorphic vector field v generating $\mathcal{F}$ (so that v induces a local trivialization of $T_{\mathcal{F}}$). Then the squared norm of v with respect to the leafwise Poincaré metric is equal to e^F, where F has values in $[-\infty, +\infty)$ and satisfies

$$\mathcal{L}_v \mathcal{L}_{\bar{v}} F = e^F$$

($\mathcal{L}_v$, $\mathcal{L}_{\bar{v}}$ are the Lie derivatives). The same holds if $p \in Sing(\mathcal{F})$, after the choice of a local vector field vanishing only at p and generating $\mathcal{F}$, the only difference being that F is not (yet) defined at p. And the same holds also if $p \in Sing(X)$, provided that we replace $T_{\mathcal{F}}$ by $T_{\mathcal{F}}^{\otimes n}$ in order to get local freeness; here there is no problem in defining F at p. The functions F are the local weights of a (singular) hermitian metric on $T_{\mathcal{F}}|_{X'}$, and by dualizing we obtain a (singular) hermitian metric on $K_{\mathcal{F}}|_{X'}$, which will be called **canonical Poincaré metric**. For the moment it is defined only on X', but below we shall see that it naturally extends to the full X. We refer to [Dem] for the basic aspects of singular hermitian metrics that we shall use.

The curvature Ω of the canonical Poincaré metric is locally given by $\frac{i}{2\pi}\partial\bar{\partial}F$, provided that it makes sense. If the local weights F are locally integrable then Ω is a well defined closed (1,1)-current on X', which represents the first Chern class of $K_{\mathcal{F}}$. One says that the curvature is **positive** if Ω is a closed positive current, i.e. the local weights F are plurisubharmonic. Note that if Ω is positive then, according to classical extension results for plurisubharmonic functions, it extends to the full X, as a closed positive current. Thus in that case the canonical Poincaré metric is everywhere defined.

We can now state our main result.

Theorem. *Let $\mathcal{F}$ be a nef foliation on a compact complex algebraic surface X. Suppose that there exists at least one hyperbolic leaf. Then the canonical Poincaré metric has positive curvature.*

Of course, if every leaf is parabolic then the leafwise Poincaré metric is identically zero and thus it is a quite uninteresting object. The classification of these totally parabolic foliations can be found in [McQ] (see also [Bru]), as a particular case of more general results concerning foliations with at least one nonalgebraic parabolic leaf (but it turns out that the existence of such a leaf implies that every leaf is parabolic).

Remark that the *existence* of a singular hermitian metric on $K_{\mathcal{F}}$ whose curvature is positive is simply a consequence of the nefness of $K_{\mathcal{F}}$ [Dem]. Our theorem can be seen as a concretization of such an existential result. An advantage is that we gain some regularity: our metric is real analytic along the leaves. The transverse regularity is certainly a major open problem. When there are no parabolic leaves one can use Brody's reparametrization lemma as in [Can] to prove that the weights F are continuous outside $Sing(\mathcal{F})$. But when there are parabolic leaves (which may be supposed algebraic, thanks to [McQ] and [Bru]) the situation seems more complicated, because Brody's lemma does not control the supports of the entire curves that it produces, and so one cannot easily prove that the weights F are continuous outside $Sing(\mathcal{F})$ and the parabolic leaves (as they should be). We shall discuss in the next section a particular case of this problem, when the parabolic leaves are contractible to normal singularities. We shall also state a result of McQuillan which does not require the contractibility but excludes the algebraic leaves.

Let us now give the outline of the proof.

Let $T \subset X''$ be a (small) embedded disc transverse to $\mathcal{F}$. For every $q \in T$ let L_q be the leaf of $\mathcal{F}$ through q and let $\tilde{L}_q$ ($\simeq$ **D** or **C**) be its universal covering. It turns out that these universal coverings glue together to a smooth complex surface U_T (called **covering tube**) which fibers over T and whose fibre over q is $\tilde{L}_q$. On each fibre of U_T we put its Poincaré metric, and evidently we have to prove that *this* leafwise (or fibrewise) Poincaré metric on U_T has positive curvature. If U_T is a Stein surface, then such a result has been already proved by Yamaguchi [Yam] and Kizuka [Kiz]. Moreover, if X were a Stein surface (instead of a projective one) then U_T would also be a Stein surface, by [Ily] and [Suz]. However, the proofs of [Ily] or [Suz] does not obviously extend to the projective case: those proofs are based on Cartan–Thullen–Oka type results ("an unramified domain of holomorphy over a Stein manifold is Stein"), and similar results are not available for domains of holomorphy over projective manifolds. We encounter here, at an easier level, the same kind of difficulties that one encounters around Shafarevich conjecture, asserting the holomorphic convexity of the universal covering of a projective manifold. Indeed, our surface U_T is something like the "leafwise universal covering" of $\mathcal{F}$.

In spite of these difficulties, we shall prove that U_T is morally Stein. We shall prove that it satisfies a convexity property which is possibly equivalent to its Steinness and which is anyway sufficient to carry out Yamaguchi–Kizuka proof of the subharmonicity of the variation of the fibrewise Poincaré metric. To do this, we will be lead to solve a sort of "Riemann–Hilbert boundary value problem" in U_T, i.e. to construct Levi-flat hypersurfaces with a given real torus as boundary [B-G] [For]. A major ingredient will be Bishop–Gromov compactness theorem for holomorphic discs of bounded area [Pan] [Iva].

The proof of the Theorem will be carried out in the next two sections. In the last section we shall discuss an application of the Theorem concerning the Kodaira dimension of foliations [Bru] [McQ], application which was our initial motivation: *the only reduced nef foliations of Kodaira dimension* $-\infty$ *are the Hilbert modular foliations*. Together with McQuillan's results, this achieves the classification of foliations of nongeneral type.

It is a pleasure to thank M. McQuillan for many stimulating conversations around Hilbert modular foliations.

2. Covering tubes

Let $\mathcal{F}$ be a nef foliation on a compact complex algebraic surface X. Let $T \subset X''$ be an embedded disc (up to the boundary) transverse to $\mathcal{F}$. The following Proposition can be found in [Ily] when X is Stein; a related result appears also in [Suz]. Here we replace the Steinness of X with the nefness of $K_{\mathcal{F}}$.

Proposition 1. *There exists a smooth complex surface* U_T *such that:*

i) U_T *admits a submersion* $P : U_T \to T$ *onto* T *with connected and simply connected fibres* $P_t = P^{-1}(t)$, *and* P *admits a holomorphic section* $s : T \to U_T$;
ii) U_T *admits an immersion* $\pi : U_T \to X'$ *such that* $\pi(P_t) = L_t$ *(the leaf of* $\mathcal{F}$ *through* t*),* $\pi(s(t)) = t$, *and* $\pi|_{P_t} : (P_t, s(t)) \to (L_t, t)$ *is the universal covering of* L_t *with basepoint* t.

(terminological clarification: if $\pi(x) \in Sing(X)$ then to say that π is an immersion at x means, by definition, that π is around x the composition of an immersion into $\mathbf{B}^2$ followed by the quotient map $\mathbf{B}^2 \to \mathbf{B}^2/\mathbf{Z}_n \subset X'$).

Proof. First of all observe that, following [Suz, §3], there is no problem in constructing a smooth complex surface V_T fibered over T which satisfies all the properties above except that its fibres are *holonomy* coverings of leaves, instead of universal ones. Simply take a holomorphic function h on a neighbourhood of T defining $\mathcal{F}$ (a so-called first integral), take its domain of holomorphy $\mathcal{D}$ over X' ($\mathcal{D}$ contains T and h extends to $\mathcal{D}$, in such a way that it becomes a first integral for the foliation lifted to $\mathcal{D}$), and finally take the union $V_T \subset \mathcal{D}$ of the connected components of regular fibres of h

intersecting $T \subset \mathcal{D}$. Note that V_T is smooth, because if $x \in Sing(X)$ then the holonomy of the local leaf through x has period equal to the order of the local covering $\mathbf{B}^2 \to \mathbf{B}^2/\mathbf{Z}_n \subset X'$ (strictly speaking, in the previous construction we should firstly take the domain of holomorphy over X'', and then to "complete" this domain by adding the points over $Sing(X)$...).

Let now γ be a smooth embedded cycle contained in a fibre H of V_T, and let γ' be a deformation of γ in a nearby fibre H'. Suppose that γ' is homotopic to zero in H', so that it bounds a disc $\Gamma' \subset H'$. As in [Ily], we want to prove that γ also bounds a disc Γ in H.

Fix on $\mathbf{C}^2$ the sup-metric and set $\Omega_\epsilon = \epsilon$-neighbourhood of

$$\{(z, w)|z = 0, |w| \leq 1\} \cup \{(z, w)|z \in [0, 1], |w| = 1\},$$

and $\hat{\Omega}_\epsilon = \epsilon$-neighbourhood of

$$\{(z, w)|z \in [0, 1], |w| \leq 1\}.$$

Note that $\hat{\Omega}_\epsilon$ is the envelope of holomorphy of Ω_ϵ. Up to slightly changing γ and γ', we have a holomorphic embedding $i : \Omega_\epsilon \to V_T$, for $\epsilon > 0$ sufficiently small, such that:

1) i maps each vertical fibre of Ω_ϵ into a fibre of V_T;
2) $i(\{z = 1, |w| = 1\}) = \gamma$, $i(\{z = 0, |w| \leq 1\}) = \Gamma'$.

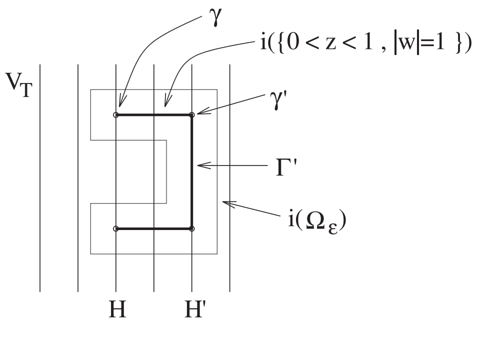

Fig. 1

Let $j : \Omega_\epsilon \to X$ be the composition of i and the immersion $V_T \xrightarrow{\pi} X' \subset X$. Because X is algebraic, the map j extends to a meromorphic map [Iva]

$$\hat{j} : \hat{\Omega}_\epsilon - - - \to X.$$

A priori, $\hat{j}$ could have indeterminacy points, in $\hat{\Omega}_\epsilon \setminus \Omega_\epsilon$, but we shall see that it is not the case, thanks to the nefness of $K_{\mathcal{F}}$.

Remark that indeterminacy points can occur only on a discrete (finite) set of vertical fibres. From the fact that $\hat{j}$ is an immersion on Ω_ϵ it follows that $\hat{j}$ is an immersion on $\hat{\Omega}_\epsilon \setminus$ {indeterminacy points}. Thus if $D_z \subset \hat{\Omega}_\epsilon$ is a vertical fibre free of indeterminacy points then $\hat{j}|_{D_z} : D_z \to X$ is an immersion, tangent to $\mathcal{F}$ because of 1) above. From the fact that $\hat{j}$ is a local biholomorphism of $\hat{\Omega}_\epsilon$ into X around each point of D_z, it follows also that $\hat{j}(D_z)$ is disjoint from $Sing(\mathcal{F})$, i.e. $\hat{j}(D_z)$ is an immersed disc inside a leaf of $\mathcal{F}$.

Lemma 1. *Let $\mathcal{F}$ be a nef foliation on a surface X and let*

$$f : \mathbf{D} \times \mathbf{D} - - - \to X$$

be a meromorphic map such that:

i) *f is an immersion outside its indeterminacy points;*
ii) *$f^*(\mathcal{F})$ is the vertical foliation on $\mathbf{D} \times \mathbf{D}$.*

Then f is holomorphic.

The proof will be given below. By this lemma, $\hat{j}$ is holomorphic and $\hat{j}(\{z = 1, |w| \leq 1\})$ is an immersed disc in the leaf $\pi(H)$, with boundary $\pi(\gamma)$. This immersed disc can be lifted to V_T, giving a disc $\Gamma \subset H$ with $\partial\Gamma = \gamma$.

More generally, let $\gamma \subset H$ be a smooth cycle, not necessarily embedded, and let again $\gamma' \subset H'$ be a small deformation in a nearby fibre. We claim that, again, if γ' is homotopic to zero in H' then γ also is homotopic to zero in H. To see this, we may suppose that γ has only double points, as well as γ' (the "same" as γ). Let $N \subset H$ be a tubular neighbourhood of γ, with ∂N smooth, and let $N' \subset H'$ be a tubular neighbourhood of γ', deformation of N. Let $K' \subset H'$ be the compact subsurface with boundary obtained by adding to N' all the connected components of $H' \setminus N'$ which are closed discs. It is easy to see (just look at the universal covering) that $\pi_1(K')$ injects into $\pi_1(H')$, hence γ' is homotopic to zero also in K'. By the previous result, the components of $\partial N'$ bounding a disc in $H' \setminus N'$ correspond to components of ∂N which also bound a disc in $H \setminus N$. We thus obtain a compact subsurface $K \subset H$, diffeomorphic to K', in which γ is homotopic to zero. In particular, γ is homotopic to zero also in H.

Now the proof of the Proposition can be completed as in [Ily, §1]: the "fibrewise" universal covering of V_T (with basepoints on T) is Hausdorff, and so it is a complex surface U_T with all the desired properties. □

Proof of Lemma 1. We may of course suppose that f has only one indeterminacy point, say the origin (0, 0), and we shall derive a contradiction.

Let $Y_0 \subset (\mathbf{D} \times \mathbf{D}) \times X$ be the graph of f, and let $Y_1 \to Y_0$ be the resolution of its singularities. The projection $h_1 : Y_1 \to \mathbf{D} \times \mathbf{D}$ is a bimeromorphic map which contracts a tree of rational curves T over (0, 0). Some of these curves may be contracted by the second projection $g_1 : Y_1 \to X$; however, no curve outside T is contracted by g_1, because f is an immersion outside (0, 0). If we contract to a point each maximal subtree $T' \subset T$ contracted by g_1, we obtain a normal surface Y equipped with a projection $g : Y \to X$ which contracts no curve, and a bimeromorphic projection $h : Y \to \mathbf{D} \times \mathbf{D}$ which contracts over $(0, 0)$ a union of rational curves $R = \cup_{j=1}^{n} R_j$ with negative definite intersection form.

Set $\mathcal{G} = g^*(\mathcal{F}) = h^*$(vertical foliation) (as usual, foliations are "saturated", i.e. $\mathcal{G}$ has only isolated singularities; note that $\mathcal{G}$ is not necessarily a foliation in our standing meaning, because Y could have complicated singularities). Remark that $\mathcal{G}$ is tangent to R. We shall compute $K_{\mathcal{G}}$ (a $\mathbf{Q}$-bundle) in two ways, using g and using h.

Because g contracts no curve, we have $g(p) \in X''$ for p outside a finite subset of Y. If such a p is also outside R then g is a local biholomorphism around p. If such a p is a generic point of R then g has possibly a ramification along R, but anyway $g|_R$ is, around p, a local biholomorphism of R into the leaf of $\mathcal{F}$ through $g(p)$. In both cases, we see that g^* realizes a natural local isomorphism between sections of $K_{\mathcal{F}}$ and sections of $K_{\mathcal{G}}$. Hence $g^*(K_{\mathcal{F}}) = K_{\mathcal{G}}$ outside a finite subset of Y, that is $g^*(K_{\mathcal{F}}) = K_{\mathcal{G}}$ tout court. In particular $K_{\mathcal{G}}$ is nef, i.e. $K_{\mathcal{G}} \cdot R_j \geq 0$ for every $j = 1, ..., n$.

On the other hand, let us consider the vector field $\frac{\partial}{\partial w}$ on $\mathbf{D} \times \mathbf{D}$, which generates the vertical foliation. Its pull-back on Y via h is a meromorphic vector field v, whose zero and polar divisors are contained in R. Hence $K_{\mathcal{G}} = \mathcal{O}_Y(\sum_{j=1}^{n} m_j R_j)$ where $m_j \in \mathbf{Z}$ is negative if v vanishes on R_j at order $-m_j$, positive if v has a pole along R_j of order m_j. Because $K_{\mathcal{G}} \cdot R_j$ is nonnegative for every j and because $(R_i \cdot R_j)_{i,j}$ is negative definite, a well-known lemma of Zariski [Zar, p. 588] affirms that $-\sum_{j=1}^{n} m_j R_j$ is effective, i.e. $m_j \leq 0$ for every j. Thus v is actually holomorphic, but this is an evident contradiction. For instance, look at the local flow of v: it preserves R and it projects on $\mathbf{D} \times \mathbf{D}$ to the local flow of $\frac{\partial}{\partial w}$, which does not fix $(0, 0)$. □

The attentive reader has certainly observed that the introduction of the intermediate holonomy covering V_T in the proof of Proposition 1 is not absolutely indispensable (even if it helps to clean the proof). However we have constructed it, following [Suz], because we think that it is a quite interesting and useful object, which could deserve more attention. In the sequel we will be concerned only with the universal covering U_T, the reason being that we want to study the Poincaré metric on the leaves and such a metric arises from the universal coverings. But non-simply connnected Riemann surfaces (like V_T-fibres) have other interesting invariants (harmonic modules...), and these invariants are lost when we pass to the universal covering. If one can prove that V_T is Stein, or at least holomorphically convex, then the works of Yamaguchi [Yam] will give information on the (subharmonic) variation of these invariants. Note, however, that there are special situations in which V_T may fail to be holomorphically convex: this happens, for instance, when the foliation has a singular point around which there exists a holomorphic irreducible first integral. In order to obtain something which could be holomorphically convex we should, at least, add to V_T some isolated points, as in Suzuki's paper [Suz].

Remark also that Proposition 1, and more specifically Lemma 1, may fail without the nefness hypothesis on $K_{\mathcal{F}}$, as the blow-up of a foliation at a regular point shows. In fact, the property expressed by Lemma 1 is really equivalent to the nefness of $K_{\mathcal{F}}$, and in this sense it sheds some light on McQuillan's work on minimal models [McQ].

The complex surface U_T will be called **covering tube**. Its base T will be frequently identified with its image $s(T) \subset U_T$ by s, and also with the unit disc $\mathbf{D}$. We shall denote by $D(z,r)$ the disc in T centered at z and of radius r. From the differentiable point of view U_T is trivial, that is diffeomorphic to $T \times \mathbf{R}^2$ [Mei]. The next result provides a weak form of holomorphic convexity of U_T.

Main Lemma. *Let $\mathcal{F}$ be a nef foliation on X and let $U_T \stackrel{P}{\to} T$ be a covering tube associated to $\mathcal{F}$. Then for every $z \in T$ there exists a finite subset $I_z \subset (0, dist(z, \partial T))$ such that for every $r \in (0, dist(z, \partial T)) \setminus I_z$ the following holds. For every compact $K \subset P^{-1}(\partial D(z,r))$ there exists a real analytic 2-torus $S \subset P^{-1}(\partial D(z,r))$ such that:*

i) *S is transverse to the fibres of $P^{-1}(\partial D(z,r)) \stackrel{P}{\to} \partial D(z,r)$ and it encloses K, i.e. for every $z' \in \partial D(z,r)$ the intersection $S \cap P^{-1}(z')$ is a circle which bounds on $P^{-1}(z')$ a disc containing $K \cap P^{-1}(z')$;*
ii) *S is in $P^{-1}(\overline{D(z,r)})$ the boundary of a real analytic Levi-flat horizontal hypersurface M_S, disjoint from T and filled by disjoint images of holomorphic sections $s_\theta : \overline{D(z,r)} \to P^{-1}(\overline{D(z,r)})$, $\theta \in \mathbf{S}^1$, with boundary values on S.*

Let us immediately do two commentaries on this statement:

a) the finite subsets I_z will be introduced along the proof, in an explicit form, due to some rather marginal and technical problems, but the statement should be true for every $r \in (0, dist(z, \partial T))$.
b) the torus S can be in fact choosen in a rather free way: that is, we will show that if $S \subset P^{-1}(\partial D(z,r))$ is *any* torus which satisfies i) plus a relatively mild condition (explained along the proof) then S bounds a Levi-flat hypersurface M_S as in ii). Hence the Main Lemma is a partial solution to the Riemann–Hilbert boundary value problem in U_T [For].

The Main Lemma will be proved in the next section. Here we firstly derive the Theorem, following [Yam] and [Kiz].

Proof of the Theorem. Let $U_T \stackrel{P}{\to} T$ be a covering tube and fix the Poincaré metric on the fibres of P. By property ii) of Proposition 1, this is the pull-back of the leafwise Poincaré metric. Take local coordinates (z, w) around $T \subset U_T$ such that $T = \{w = 0\}$ and $P(z, w) = z$, and let F be the local weight of the metric in these coordinates. Due to the arbitrarity of $T \subset X''$, we have only to prove that the restriction $F(\cdot, 0)$ of F to T is subharmonic, i.e. it satisfies the submean inequality (as already observed, the upper semicontinuity is immediate from the fact that a compact disc in a fibre of P can be shifted in nearby fibres [Suz, §2]).

Take $z \in T$ and r as in the Main Lemma. For every $S \subset P^{-1}(\partial D(z, r))$ as there, let $\Omega_S \subset P^{-1}(\overline{D(z, r)})$ be the open subset bounded by M_S and containing $\overline{D(z, r)} \subset U_T$. The fibres of $\Omega_S \stackrel{P}{\to} \overline{D(z, r)}$ are discs, on which we put the Poincaré metric. Denote by F_S the corresponding local weight. Because $\partial\Omega_S = M_S$ is Levi-flat, the computation of [Yam] (see also [Kiz]) shows that F_S is plurisubharmonic, and therefore

$$F_S(z, 0) \leq \frac{1}{2\pi} \int_0^{2\pi} F_S(z + re^{i\theta}, 0) d\theta.$$

On the other hand, the fibres of Ω_S being inside the fibres of U_T, we also have

$$F(z, 0) \leq F_S(z, 0).$$

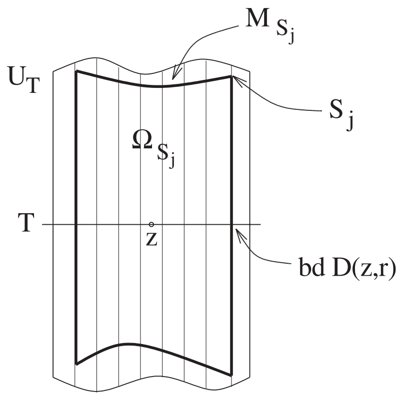

Fig. 2

By varying the compact set K in the Main Lemma, we obtain a sequence of tori $\{S_j\}$ such that for every $z' \in \partial D(z, r)$ the sequence $\{\Omega_{S_j} \cap P^{-1}(z')\}$ is increasing and exhausting in $P^{-1}(z')$. Thus the corresponding weights F_{S_j}

decreasingly converge to F on $\partial D(z, r)$. Therefore

$$F(z, 0) \leq \lim_{j\to+\infty} F_{S_j}(z, 0) \leq \lim_{j\to+\infty} \frac{1}{2\pi} \int_0^{2\pi} F_{S_j}(z + re^{i\theta}, 0)d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} F(z + re^{i\theta}, 0)d\theta$$

which proves the subharmonicity of $F(\cdot, 0)$ (for the arbitrarity of $z \in T$ and the almost arbitrarity of r), unless $F \equiv -\infty$. In this last case, a connectivity argument shows that if some local weight of the leafwise Poincaré metric is $-\infty$ then every local weight is $-\infty$ (a psh function cannot be equal to $-\infty$ on a nonempty open set), and hence all the leaves of $\mathcal{F}$ are parabolic. This completes the proof. □

The property expressed by the Main Lemma seems quite close to the Steinness of U_T. For instance, suppose that all the fibres of U_T are hyperbolic, and that the metric is there continuous (a nonvoid case, see below). Then the function $\phi : U_T \to \mathbf{R}^+$ defined by "$\phi(q)$ = squared distance between q and its projection on $T \subset U_T$" is continuous, psh (by our Theorem and a relatively subtle computation that we do not reproduce here) and exhaustive in the vertical direction. If $\psi : T \to \mathbf{R}^+$ is a smooth strictly subharmonic exhaustive function, then $\hat{\phi} = \phi + \psi \circ P$ is a continuous psh exhaustive function on U_T. This is not yet sufficient to conclude that U_T is Stein, unless the metric (and hence ϕ) is of class C^2, in which case one immediately sees that $\hat{\phi}$ is *strictly* psh. However, we may suppose without loss of generality, thanks to the classification results of [McQ] and the present paper, that $\mathcal{F}$ is a general type foliation (otherwise $U_T \simeq T \times \mathbf{D}$) and therefore $K_{\mathcal{F}}$ admits a *smooth* hermitian metric whose curvature is strictly positive outside the parabolic (rational) leaves [McQ]. This metric induces a metric on the fibres of U_T, and the corresponding squared distance $\phi' : U_T \to \mathbf{R}^+$ is smooth and strictly psh (but *not* exhaustive in the vertical direction, because the singularities of the foliation allow the metric on the fibres to be incomplete). Then the function $\phi + \phi' + \psi \circ P$ is continuous, strictly psh and exhaustive on U_T, which is consequently Stein.

Let us conclude this section with two results concerning the regularity of the leafwise Poincaré metric, which are independent from the theorem but whose proofs are related to the above Proposition 1. Recall that the existence of a hyperbolic leaf implies that the parabolic leaves are algebraic [McQ] [Bru], so that the regularity problem makes sense only if

$$\mathcal{P} = \text{closure}\{\text{parabolic leaves}\} \subset X$$

is a (possibly empty) algebraic curve. Moreover, everything is quite trivial if the foliation is a fibration, so that we may restrict to the case where there is a finite number of algebraic leaves, or equivalently

$$\mathcal{A} = \text{closure}\{\text{algebraic leaves}\} \subset X$$

is a (possibly empty) algebraic curve. The first result is an adaptation of [Can] to our singular context.

Proposition 2. *Let $\mathcal{F}$ be a nef foliation on X and suppose that the closure of the parabolic leaves $\mathcal{P}$ is an algebraic curve, each connected component of which is contractible to a normal singularity. Then the leafwise Poincaré metric is continuous on $X \setminus (\mathcal{P} \cup Sing(\mathcal{F}))$.*

Proof. Let $\hat{X}$ be the normal surface obtained by contracting $\mathcal{P}$, and let $Z \subset \hat{X}$ be the image of $\mathcal{P} \cup Sing(\mathcal{F})$ in $\hat{X}$ (a finite set). We shall work on $\hat{X}$ instead of X. We shall denote by $\hat{\mathcal{F}}$ the image of $\mathcal{F}$ in $\hat{X}$; note that the leaves of $\hat{\mathcal{F}}$ outside Z are the same as the leaves of $\mathcal{F}$ outside $\mathcal{P} \cup Sing(\mathcal{F})$. Take $p \in \hat{X} \setminus Z$ and a sequence $\{p_n\} \subset \hat{X} \setminus Z$ converging to p. For each n, let $f_n : \mathbf{D} \to L_{p_n}$ be a holomorphic map sending the unit disc into the leaf through p_n, and sending 0 to p_n. We need to prove that a subsequence of $\{f_n\}$ converges uniformly on compact subsets to a holomorphic map $f : \mathbf{D} \to L_p$, $f(0) = p$: this will give the lower semicontinuity at p of the local weight F around p, and hence its continuity.

Fix an hermitian metric ω on $\hat{X}$ (by definition, around a singularity of $\hat{X}$ this is the restriction of an hermitian metric defined on an euclidean space in which $\hat{X}$ locally embeds). For every compact $K \subset \mathbf{D}$ consider the set $I_K = \{\|f_n'(z)\|_\omega | z \in K, n \in \mathbf{N}^+\}$; we claim that it is bounded. Indeed, in the opposite case we can find a subsequence $n_j \to +\infty$ and $z_j \in K$ such that $\|f_{n_j}'(z_j)\|_\omega \to +\infty$, and by composing with automorphisms of $\mathbf{D}$ (of bounded norm) sending 0 to z_j we can find another sequence $g_j : \mathbf{D} \to \hat{X}$ such that $g_j(\mathbf{D}) = f_{n_j}(\mathbf{D})$ and $\|g_j'(0)\|_\omega \to +\infty$. By Brody's lemma [Kob], we can construct a nonconstant entire curve $h : \mathbf{C} \to \hat{X}$, uniform limit on compact sets of suitable reparametrizations $h_j : \mathbf{D}(r_j) \to \hat{X}$ of g_j (where $\mathbf{D}(r_j)$ is the disc of radius r_j and $r_j \to +\infty$). Being $h_j(\mathbf{D}(r_j)) = g_j(\mathbf{D}) = f_{n_j}(\mathbf{D}) \subset L_{p_{n_j}}$, h is clearly tangent to the foliation, but we assert that it is also entirely contained in a leaf, i.e. $h(\mathbf{C}) \subset \hat{X} \setminus Z$. To see this, note that if $D \subset \hat{X}$ is a small disc through $q \in Z$ tangent to the foliation, then its strict transform $\tilde{D} \subset X$ is a disc through a singularity $\tilde{q} \in Sing(\mathcal{F})$, tangent to $\mathcal{F}$. If $D_n \subset \hat{X} \setminus Z$ are discs in leaves which approximate D, then $\tilde{D}_n$ could fail to approximate $\tilde{D}$, but anyway $\partial\tilde{D}_n$ still approximate $\partial\tilde{D}$. We then arrive rapidly to a contradiction with Proposition 1 (see especially its proof, giving a sort of semicontinuity of the fundamental groups of the leaves). Hence $h(\mathbf{C})$ is inside a leaf, and this is impossible because on $\hat{X}$ no leaf is parabolic.

The boundedness of I_K for every $K \subset \mathbf{D}$ implies that a subsequence $\{f_{n_j}\}$ converges uniformly on compact sets to some $f : \mathbf{D} \to \hat{X}$, $f(0) = p$. As in the case of h, one verifies that $f(\mathbf{D}) \cap Z = \emptyset$, that is $f(\mathbf{D}) \subset L_p$. □

The second result has been proved by McQuillan in [McQ], and is based on much more sophisticated tools. This result will be used in the last section of this paper.

Proposition 3 [McQ]. *Let $\mathcal{F}$ be a nef foliation on X and suppose that the closure of the algebraic leaves $\mathcal{A}$ is an algebraic curve, containing $\mathcal{P}$. Then the leafwise Poincaré metric is continuous on $X \setminus (\mathcal{A} \cup Sing(\mathcal{F}))$.*

Sketch of the proof. It is not difficult to reduce the statement to the case in which all the singularities of the foliation are reduced, in Seidenberg's sense [Bru], by analysing the changes of the metric under blow-up and blow-down. Also, we may suppose that the Kodaira dimension of $\mathcal{F}$, $kod(\mathcal{F})$, is not 0 nor 1, because in those cases we already have a full classification and the statement becomes trivially verifiable [McQ] [Bru]. Then the set of $\mathcal{F}$-invariant algebraic curves over which $K_{\mathcal{F}}$ has zero degree is contractible to normal singularities. We still denote by $(X, \mathcal{F})$ the result of this contraction; thus $K_{\mathcal{F}} \cdot C > 0$ for every $\mathcal{F}$-invariant algebraic curve C.

Take $p \in X \setminus (\mathcal{A} \cup Sing(\mathcal{F}))$, $p_n \to p$, $f_n : \mathbf{D} \to X$ sending 0 to p_n and $\mathbf{D}$ into the leaf L_{p_n}. As in the previous proposition, we need to show that $\{f_n\}$ admits a convergent subsequence. To do this, it is sufficient to prove that, for every $r < 1$, the "Nevanlinna degrees" $deg_r(f_n) = \int_0^r \frac{dt}{t} \int_{\mathbf{D}(t)} f_n^* \omega$ are uniformly bounded by some constant $c(r)$. Indeed, such a bound implies that $\{f_n|_{\mathbf{D}(r)}\}$ has a subsequence which converges (uniformly on compact sets) outside a finite subset of $\mathbf{D}(r)$, cfr. [Pan], but then one proves that such a finite subset is in fact empty: by Proposition 1, the convergence over a circle $\gamma \subset \mathbf{D}(r)$ implies the convergence over the full disk $\Gamma \subset \mathbf{D}(r)$ bounded by γ.

Therefore, we assume by contradiction that $\{deg_r(f_n)\}$ is unbounded, for some $r < 1$. Then McQuillan constructs from $\{f_n\}$ an $\mathcal{F}$-invariant closed positive current Φ, of bidegree (1,1), which can be thought as the "current" counterpart to Brody's lemma (one could guess that Φ is the integration current over the Brody's entire curve, but probably this is not always the case because of the Brody's reparametrizations). Basically, Φ is given by $\Phi(\beta) = \lim_{k \to +\infty} \int_0^R \frac{dt}{t} \int_{\mathbf{D}(t)} f_{n_k}^* \beta \big/ \int_0^R \frac{dt}{t} \int_{\mathbf{D}(t)} f_{n_k}^* \omega$, for suitable $R \in (r, 1)$ and $n_k \to +\infty$.

The advantage of this current, with respect to the Brody's entire curve, is that one (McQuillan) is able to prove the "tautological inequality" $c_1(K_{\mathcal{F}}) \cdot [\Phi] \leq 0$. It is here that the hypothesis $p \notin \mathcal{A}$ is used. The strict positivity of $K_{\mathcal{F}}$ over $\mathcal{F}$-invariant curves implies that Φ has zero Lelong number outside a countable (finite) set, in particular $[\Phi]$ is nef. Hodge index theorem gives $c_1(K_{\mathcal{F}}) = \lambda[\Phi]$ for some positive λ and $c_1^2(K_{\mathcal{F}}) = 0$, so that $kod(\mathcal{F})$ cannot be 2 and it is $-\infty$. On the other side, an index theorem for $\mathcal{F}$-invariant currents [Bru] gives $c_1(N_{\mathcal{F}}) \cdot [\Phi] \geq 0$, i.e. $c_1(N_{\mathcal{F}}) \cdot c_1(K_{\mathcal{F}}) \geq 0$. But then Riemann-Roch gives nontrivial sections of $K_{\mathcal{F}}$, against $kod(\mathcal{F}) = -\infty$.

□

One expects continuity also along hyperbolic algebraic leaves. In fact, still according to McQuillan (personnal communication) this is indeed true, and the proof will appear in a new version of [McQ].

3. Proof of the Main Lemma

To begin with, we fix a real analytic Kähler metric ω on X, in the orbifold sense: around a point in $Sing(X)$, ω is the cyclic quotient of a Kähler metric on the local covering $\mathbf{B}^2$. (Such a metric exists by the following standard reasoning. We start with a Kähler metric on $X \setminus Sing(X)$, arising for instance from the resolution of X. On a neighbourhood $U = \mathbf{B}^2/\mathbf{Z}_n$ of $p \in Sing(X)$ we thus have a metric outside p, which can be lifted to $\mathbf{B}^2 \setminus \{0\}$. This lifting can be extended to $\mathbf{B}^2$ as a closed positive current, and this extension can be smoothly regularized without changing it outside $\frac{1}{2}\mathbf{B}^2$ and in an equivariant way. Hence this regularization can be projected to U and glued with the old metric. Finally we C^ω-regularize).

Let now $U_T \stackrel{P}{\rightarrow} T$ be a covering tube, with immersion $\pi : U_T \rightarrow X$ mapping P-fibres onto leaves. We have a real analytic Kähler metric $\pi^*(\omega)$ on U_T, which however we shall denote again by ω.

If $\tilde{L} \subset U_T$ is a fibre of P, projecting onto the leaf L, and if L is not algebraic (i.e., not contained in an algebraic curve) then the volume of $\tilde{L}$, $vol(\tilde{L}) = \int_{\tilde{L}} \omega$, is certainly infinite: a nonalgebraic leaf must accumulate on a regular point of the foliation, and the local product structure of the foliation around such a point shows that $vol(L) = +\infty$ and consequently $vol(\tilde{L}) = +\infty$. If L is algebraic and $\pi_1(L)$ is infinite, then obviously $vol(\tilde{L}) = +\infty$. We thus see that the only leaves whose universal covering has finite volume are algebraic leaves whose closure is a rational curve containing only one singularity of $\mathcal{F}$, and at most one singularity of X. If the set of these leaves is infinite then $\mathcal{F}$ has a rational first integral (Jouanolou's theorem, see [Bru, p. 84]) and it is something like a pencil of lines. In this case one easily verifies that $U_T \simeq T \times \mathbf{C}$ and so everything becomes trivial. Therefore, we shall suppose from now on that $\mathcal{F}$ has only a finite number of such leaves. In particular, T cuts the set of these leaves at a finite set of points, so that if we define

$$T_0 = \{z \in T | vol(P^{-1}(z)) = +\infty\}$$

then $card\ (T \setminus T_0) < +\infty$. And if we define, for every $z \in T$,

$$I_z = \{r \in (0, dist(z, \partial T)) | \partial D(z, r) \not\subset T_0\}$$

then I_z is a finite set, with $card\ I_z \leq card\ (T \setminus T_0)$.

Take now $z \in T$ and $r \in (0, dist(z, \partial T)) \setminus I_z$, so that $\partial D(z, r) \subset T_0$. In the following z and r will be fixed, so to ease the notation we shall forget them and we shall write D instead of $D(z, r)$. Take a compact set $K \subset P^{-1}(\partial D)$. Without loss of generality, for the purposes of the Main Lemma, we may assume that K contains all the points of T ($\subset U_T$) over ∂D. If $z \in \partial D$ and if $S \subset P^{-1}(\partial D)$ is a torus enclosing K, we shall set $K_z = K \cap P^{-1}(z)$, $T_z = T \cap P^{-1}(z) \in K_z$, $S_z = S \cap P^{-1}(z)$ (a circle enclosing K_z), $S_z^0 =$ bounded component of $P^{-1}(z) \setminus S_z$ (a disc containing K_z) (such a notation, in fact, will be used even if S does not enclose K).

Lemma 2. *There exists a real analytic torus $S \subset P^{-1}(\partial D)$ transverse to the fibres of $P^{-1}(\partial D) \xrightarrow{P} \partial D$, enclosing K and such that the function*

$$z \mapsto vol\left(S_z^0\right), \qquad z \in \partial D$$

is constant.

Proof. Obviously we may find a C^∞-smooth torus $\tilde{S}$ with the required properties, because $vol(P^{-1}(z)) = +\infty$ for every $z \in \partial D$. We approximate $\tilde{S}$ by a real analytic torus $\hat{S}$; the function $z \mapsto vol(\hat{S}_z^0)$ will be real analytic and almost constant (say, close to a constant in the C^∞-topology). Observe now that on each fibre $\hat{S}_z^0$ we have a canonically defined real analytic fibration by circles, singular at T_z: the constant-radius circles centered at T_z and with respect to the hyperbolic metric on $\hat{S}_z^0$. By the real analyticity of $\hat{S}$ and ω, the circles which enclose discs of the same ω-volume glue together into real analytic tori. We thus obtain (provided that $\hat{S}$ is sufficiently close to $\tilde{S}$) a real analytic torus S close to $\hat{S}$ and with $z \mapsto vol(S_z^0)$ constant. □

Our objective is to construct a real analytic Levi-flat horizontal hypersurface $M_S \subset P^{-1}(\overline{D})$ with $\partial M_S = S$, where S is *any* torus provided by the previous lemma. As frequent in these types of problems ([B-G], [For], ...) we shall use the continuity method, and hence we immediately note the following fact, whose proof is already contained in the proof of Lemma 2.

Lemma 3. *If $S \subset P^{-1}(\partial D)$ is as in Lemma 2, then there exists a real analytic family of tori $S_t \subset P^{-1}(\partial D)$, $t \in (0, 1]$, such that:*

i) *every S_t is transverse to the fibres of $P^{-1}(\partial D) \xrightarrow{P} \partial D$ and the function $z \mapsto vol((S_t)_z^0)$ is constant;*
ii) *every S_t encloses $T \cap P^{-1}(\partial D)$ and S_t degenerates to the circle $T \cap P^{-1}(\partial D)$ as $t \to 0$;*
iii) *$S_1 = S$.*

Here by "real analytic family of tori" we mean that there exists a real analytic *embedding* $f : \partial D \times \overline{\mathbf{D}} \to P^{-1}(\partial D)$ such that $S_t = f(\partial D \times \{|z| = t\})$ for every $t \in (0, 1]$, and $T \cap P^{-1}(\partial D) = f(\partial D \times \{0\})$. A similar terminology will be implicitely employed below, concerning Levi-flats.

Take $t_0 \in (0, 1]$. We will say that $\{S_t\}_{t \in (0, t_0]}$ has a **Levi-flat extension** if there exists a real analytic family of Levi-flat hypersurfaces $\{M_t\}_{t \in (0, t_0]}$ in $P^{-1}(\overline{D})$ such that:

i) $\partial M_t = S_t$ for every $t \in (0, t_0]$;
ii) M_t is filled by disjoint holomorphic discs which are images of holomorphic sections of P over $\overline{D}$ with boundary values on S_t;
iii) M_t degenerates to the disc $T \cap P^{-1}(\overline{D})$ as $t \to 0$.

A similar definition is given for the open family $\{S_t\}_{t\in(0,t_0)}$.

Our aim is to prove that $\{S_t\}_{t\in(0,1]}$ has a Levi-flat extension. Observe that for t_0 sufficiently small $\{S_t\}_{t\in(0,t_0]}$ certainly have a Levi-flat extension: a neighbourhood of $T \cap P^{-1}(\overline{D})$ in U_T can be mapped biholomorphically onto a neighbourhood of $\{|z| \le 1, w = 0\}$ in $\mathbf{C}^2$, in such a way that P-fibres are mapped into verticals, so that we are reduced to [For] (in a very special and relatively easy local case). By analyticity, a Levi-flat extension $\{M_t\}_{t\in(0,t_0]}$ can be analytically prolonged to a family of Levi-flats $\{M_t\}_{t\in(0,t_1)}$ for some $t_1 > t_0$, and then $\{M_t\}_{t\in(0,t_1)}$ will be, of course, a Levi-flat extension of $\{S_t\}_{t\in(0,t_1)}$. In other words, we need to prove only the following: *given $t_0 \in (0, 1]$ such that $\{S_t\}_{t\in(0,t_0)}$ has a Levi-flat extension, then also $\{S_t\}_{t\in(0,t_0]}$ has a Levi-flat extension.*

If $\{M_t\}$ is a Levi-flat extension of $\{S_t\}$, denote by Ω_t the open subset of $P^{-1}(\overline{D})$ bounded by M_t and containing $T \cap P^{-1}(\overline{D})$. Thus Ω_t is foliated, in a real analytic way, by holomorphic discs, images of holomorphic sections over $\overline{D}$, for which we have the following strong form of uniqueness: if $s_0 : \overline{D} \to \Omega_t$ is a section with $s_0(\partial D) \subset S_{t'}$, for some $t' < t$, and with $s_0(\overline{D}) \cap T = \emptyset$, then $s_0(\overline{D}) \subset M_{t'}$. This is just a consequence of the maximum principle, saying that if $s_0(D) \cap M_{t''} \neq \emptyset$ then $s_0(D) \cap M_{t''+\epsilon} \neq \emptyset$ for every $\epsilon \in \mathbf{R}$ sufficiently small, positive or negative.

In order to construct a Levi-flat extension up to t_0 we need an "a priori estimate" for the volume of discs filling M_t, $t < t_0$.

Lemma 4. *Let $\{M_t\}_{t\in(0,t_0)}$ be a Levi-flat extension of $\{S_t\}_{t\in(0,t_0)}$. Then there exists a constant $C > 0$ such that*

$$vol(D_t) \le C$$

for every holomorphic disc $D_t \subset M_t$, $\partial D_t \subset S_t$, and for every $t \in (0, t_0)$.

Proof. On a neighbourhood of S_{t_0} in $P^{-1}(\partial D)$ we may choose smooth coordinates $(\theta, \phi, r) \in (\mathbf{R}/\mathbf{Z})^2 \times \mathbf{R}$ such that $S_t = \{r = t_0 - t\}$ for every $t \le t_0$ (t close to t_0) and $\{\theta = constant\}$ are the fibres of P. For every $t < t_0$ and every disc $D_t \subset M_t$, its boundary $\partial D_t \subset S_t$ defines a homology class on S_t which is in fact constant (after the evident identifications $H_1(S_t, \mathbf{Z}) \simeq H_1(S_{t_0}, \mathbf{Z})$), by continuity. We may therefore suppose, modulo a twisting of (θ, ϕ), that this class is the same as the class of $\{\phi = 0\}$. Hence ∂D_t (which is a section over ∂D) will have an equation

$$\phi = f(\theta)\,, \qquad r = R$$

for some function $f : \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ of degree zero and for some constant $R\ (= t_0 - t)$.

Let now λ be a primitive of ω on U_T, which exists because $H^2(U_T, \mathbf{R})=0$, and note that

$$vol(D_t) = \int_{D_t} \omega = \int_{\partial D_t} \lambda.$$

The restriction of λ to the tori S_t, $t \leq t_0$ close to t_0, is expressed by $a(\theta, \phi, r)d\theta + b(\theta, \phi, r)d\phi$. The property i) of Lemma 3 means that $\int_0^1 b(\theta, \phi, r)d\phi = B(r)$ does *not* depend on θ, but only on r. In other words, we may write $b(\theta, \phi, r) = B(r) + \frac{\partial c}{\partial \phi}(\theta, \phi, r)$ for some function c, and hence

$$\lambda|_{S_t} = A(\theta, \phi, r)d\theta + B(r)d\phi + dc$$

where $dc = \frac{\partial c}{\partial \phi}d\phi + \frac{\partial c}{\partial \theta}d\theta$, $A = a - \frac{\partial c}{\partial \theta}$. When we integrate over ∂D_t, the exact term dc gives no contribution, hence

$$vol(D_t) = \int_{\phi = f(\theta)} A(\theta, \phi, R)d\theta + \int_{\phi = f(\theta)} B(R)d\phi.$$

The second integral is zero, because deg $f = 0$ (and $B(R)$ independs on θ!); the first integral is uniformly bounded because $A(\theta, \phi, r)$ is defined up to $r = 0$. □

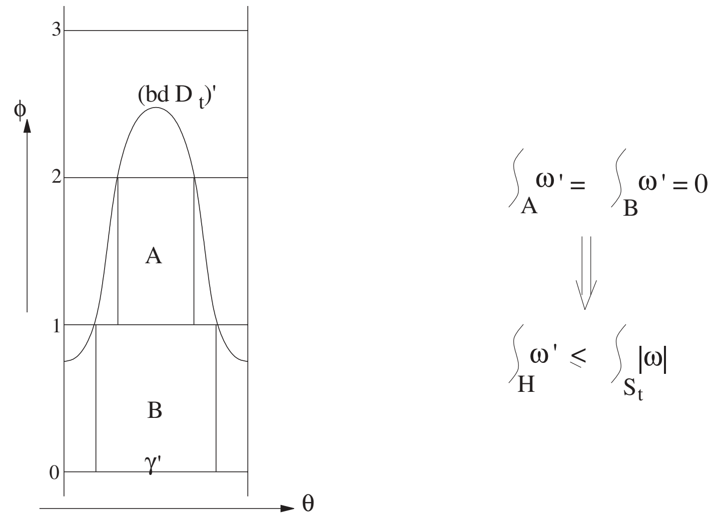

Fig. 3

Remark 1. A priori, the function f appearing in this proof could have a very large oscillation; without condition i) of Lemma 3, we would obtain an integral $\int_{\phi = f(\theta)} B(\theta, R)d\phi$ which could not be estimated in a simple way; it is however permitted to suspect that Lemma 3 is just a technical device, and that Lemma 4 holds even without the constant-volume hypothesis. The reader familiar with [B-G] and related papers has certainly recognized in the previous proof a variation on the following traditional argument: if $\overline{\mathbf{D}} \xrightarrow{i} \mathbf{C}^2$ is an embedding with $i(\partial\mathbf{D}) \subset S = $ 2-sphere, then the volume of $i(\mathbf{D})$ is bounded by the "volume" of S, because $i(\partial\mathbf{D})$ bounds a disc Γ on S and $vol(i(\mathbf{D})) = \int_\Gamma \omega \leq \int_S |\omega| = vol(S)$. In our case $\partial D_t \subset S_t$ does not bound, but on the vertical cyclic covering $S'_t \to S_t$ (universal covering of the ϕ-factor) a lifting $(\partial D_t)'$ of ∂D_t and a suitable lifting γ' of $\gamma = \{\phi = 0\}$ bound an annulus H. The geometrical meaning of our computations is that the difference $\int_{\partial D_t} \lambda - \int_\gamma \lambda = \int_H \omega'$ is actually bounded by $\int_{S_t} |\omega|$.

The previous lemma is not yet sufficient to ensure the convergence of the discs $D_t \subset M_t$ as $t \to t_0$, because U_T is noncompact. But recall that we have an immersion $\pi : U_T \to X$. The images $D'_t = \pi(D_t)$ are immersed discs with boundaries on the immersed tori $S'_t = \pi(S_t)$, and $vol(D'_t) = vol(D_t) \leq C$. According to Bishop–Gromov compactness theorem [Pan], given any sequence $\{D_{t_n} \subset M_{t_n}\}_{n\in\mathbf{N}}$ with $t_n \to t_0$, we may extract a subsequence, which we will denote by $\{D_n\}$, such that D'_n converges as $n \to +\infty$ to a "disc with bubbles" $D'_\infty \cup B_1 \cup \ldots \cup B_k$, where $D'_\infty \subset X$ is a disc with boundary on S'_{t_0} and each $B_j \subset X$ is either a rational curve or a disc with boundary on S'_{t_0}. We refer to [Pan] (see also [Iva]) for the precise notion of convergence involved here. Let us only say that, if $s_n : \overline{D} \to U_T$ is the section whose image is $\overline{D}_n$ and if $s'_n = \pi \circ s_n : \overline{D} \to X$, then there is a holomorphic map $s'_\infty : \overline{D} \to X$ and a finite set $F \subset \overline{D}$ such that $s'_\infty(\overline{D}) = \overline{D'_\infty}$ and $s'_n|_{\overline{D}\setminus F}$ converges to $s'_\infty|_{\overline{D}\setminus F}$ uniformly on compact subsets of $\overline{D} \setminus F$, as $n \to +\infty$. The set F is "where bubbles birth".

Lemma 5. *There exists a section $s_\infty : \overline{D} \setminus F \to U_T$ such that $s'_\infty|_{\overline{D}\setminus F} = \pi \circ s_\infty$ and $s_n|_{\overline{D}\setminus F}$ converges to s_∞ uniformly on compact subsets of $\overline{D} \setminus F$. Moreover $F \subset D$ (i.e. there are no boundary bubbles).*

Proof. Because $\partial D'_\infty \subset S'_{t_0} = \pi(S_{t_0})$, a tubular neighbourhood of $\partial D'_\infty$ in $\overline{D'_\infty}$ can be lifted to U_T, giving an annulus $A \subset P^{-1}(\overline{D})$ with a boundary component $a \subset S_{t_0}$. This boundary component is approximated, outside a finite set, by the curves $s_n(\partial D)$. In particular, a is not a vertical circle of S_{t_0}, and so A is not contained in $P^{-1}(\partial D)$. By the maximum principle we have $A \cap P^{-1}(\partial D) = a$. Hence A is the image of a section s_∞ over some neighbourhood of ∂D in $\overline{D}$, and s_∞ is holomorphic up to the boundary by Schwarz reflection principle. Note that $s_\infty(q)$ is defined for every $q \in \partial D$, even if $q \in F$.

This shows also that $D'_\infty = s'_\infty(D)$ is not entirely tangent to $\mathcal{F}$, and is even transverse to $\mathcal{F}$ along $\partial D'_\infty$. Let $q \in \overline{D} \setminus F$, so that $s'_n \to s'_\infty$ uniformly around q. From the fact that s'_n is an immersion transverse to $\mathcal{F}$, it follows that also s'_∞ is, around q, an immersion transverse to $\mathcal{F}$, by Rouché principle: points of nontransversality of s'_∞ are isolated and hence they persist under small deformations of s'_∞. Thus, if $D_q \subset \overline{D} \setminus F$ is a small disc around q, its image $s'_\infty(D_q) \subset X$ can be lifted to U_T, as a disc transverse to P-fibres and approximated by the discs $s_n(D_q)$. This is the image of D_q by the desired extension $s_\infty : \overline{D} \setminus F \to U_T$.

Suppose now, by contradiction, that there is a point $q \in F \cap \partial D$. Then, around $s_\infty(q)$, the curves $s_n(\partial D)$ converge to $s_\infty(\partial D) \cup b$, where b is the vertical circle of S_{t_0} over q. This b is mapped by π to the boundary of some bubble B_j, which therefore must be entirely tangent to $\mathcal{F}$. One of the two components of $P^{-1}(q) \setminus b$ is mapped by π to B_j. This is not possible for the unbounded component of $P^{-1}(q) \setminus b$: its volume is $+\infty$ and so it cannot be approximated by the discs D_n, whose volume is uniformly bounded. This is not possible also for the bounded component of $P^{-1}(q) \setminus b$: a disc D_n lies inside a Levi-flat M_n, and any other Levi-flat M_t below M_n constitutes a “lower barrier” which prevents the approximation of D_n to $(S_t)^0_q$. □

We now fix a (real) vector field v on $P^{-1}(\partial D)$ which points inward the tori S_t and is tangent to the P-fibres.

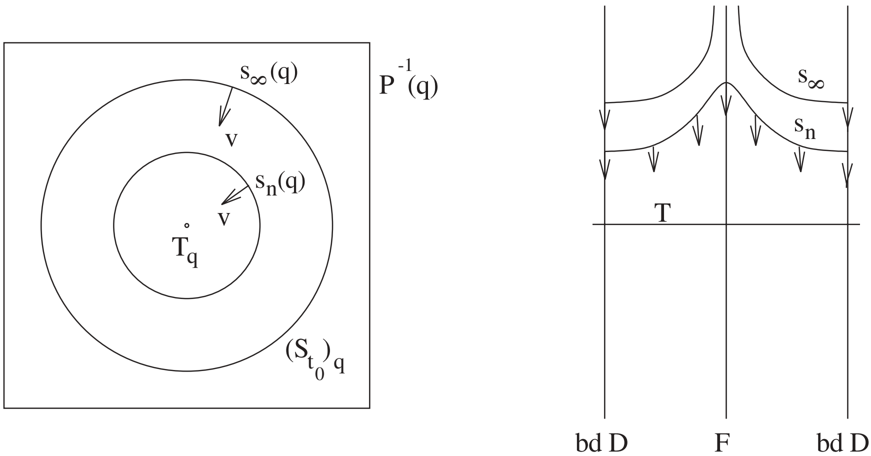

Fig. 4

This vector field, restricted to ∂D_n, gives a trivialization of $\pi^*(T_\mathcal{F})$ along ∂D_n, i.e. a trivialization of $(s'_n)^*(T_\mathcal{F})$ along ∂D. We may therefore compute $(s'_n)^*(T_\mathcal{F}) \cdot D$, the degree of $(s'_n)^*(T_\mathcal{F})$ on D. It is easy to see that $v|_{\partial D_n}$ can be extended to D_n as a nonvanishing vector field tangent to the P-fibres: just use the Levi-flat $M_n \supset D_n$. Hence that degree is zero, and dually we have

$$(s'_n)^*(K_\mathcal{F}) \cdot D = 0.$$

We may also compute the degree of $(s'_\infty)^*(K_\mathcal{F})$ on D, still using the boundary trivialization given by $v|_{s_\infty(\partial D)}$. Because s_∞ is not (yet) defined over F, we can no more conclude that this degree is zero.

Lemma 6.

$$(s'_\infty)^*(K_{\mathcal{F}}) \cdot D \geq 0$$

with strict inequality if $F \neq \emptyset$.

Proof. Let us denote by N^*_∞ the conormal sheaf of $s'_\infty : D \to X$, i.e. the kernel of the natural restriction map $(s'_\infty)^*(\Omega^1_X) \to \Omega^1_D$. If β is a local holomorphic section of N^*_∞ (arising from a 1-form on X) and w is a local holomorphic section of $(s'_\infty)^*(T_{\mathcal{F}})$ (arising from a vector field on X tangent to $\mathcal{F}$), we may evaluate the pairing $\beta(w)$. This gives a natural morphism $N^*_\infty \otimes (s'_\infty)^*(T_{\mathcal{F}}) \to \mathcal{O}_D$, i.e. a morphism $N^*_\infty \to (s'_\infty)^*(K_{\mathcal{F}})$, which vanishes exactly over the points where s'_∞ is not transverse to $\mathcal{F}$. Moreover, $N^*_\infty|_{\partial D} \simeq (s'_\infty)^*(K_{\mathcal{F}})|_{\partial D}$, by the transversality of s'_∞ to $\mathcal{F}$ along ∂D, and we may compute the degree $N^*_\infty \cdot D$ by using the same trivialization along ∂D as for $(s'_\infty)^*(K_{\mathcal{F}})$. By the previous argument, we have

$$(s'_\infty)^*(K_{\mathcal{F}}) \cdot D \geq N^*_\infty \cdot D$$

(this may be compared with the formula in [Bru, p. 23]: the difference $(s'_\infty)^*(K_{\mathcal{F}}) \cdot D - N^*_\infty \cdot D$ is nothing but the sum of indices which count the tangency points of s'_∞ with $\mathcal{F}$).

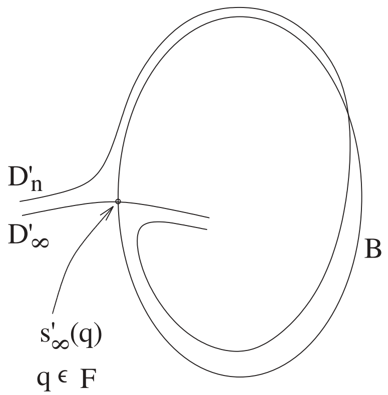

Fig. 5

We are therefore reduced to the following claim: $N^*_\infty \cdot D \geq 0$, with strict inequality if $F \neq \emptyset$. Remark that, if N^*_n is the conormal sheaf of s'_n, we already have $N^*_n \cdot D = (s'_n)^*(K_{\mathcal{F}}) \cdot D = 0$. Let us work in X, where things are geometrically clearer. We have a sequence of discs D'_n which converges to $D'_\infty \cup B$, where B is a union of rational curves (possibly with multiplicities). Then $D'_n \cdot D'_n = -N^*_n \cdot D = 0$, $D'_\infty \cdot D'_\infty = -N^*_\infty \cdot D$ (selfintersections are computed by using the boundary trivializations derived from v, and by neglecting the possible noninjectivity of s'_n and s'_∞). Because D'_n is homologous to $D'_\infty + B$ rel boundary, we obtain

$$0 = D'_n \cdot D'_n = (D'_\infty + B) \cdot (D'_\infty + B) = D'_\infty \cdot D'_\infty + D'_\infty \cdot B + D'_n \cdot B$$

and so

$$D'_\infty \cdot D'_\infty \leq 0$$

because $D'_\infty \cdot B \geq 0$, $D'_n \cdot B \geq 0$. Moreover the inequality becomes strict if $F \neq \emptyset$ (i.e. $B \neq \emptyset$), because in that case $D'_\infty \cdot B > 0$.

This completes the proof. Note that this last fact ($D'_\infty \cdot D'_\infty \leq 0$, < 0 if $B \neq \emptyset$) is simply a variation on "Zariski lemma" for singular fibres of fibrations [BPV, p. 90]. □

On the other side, the difference $(s'_n)^*(K_{\mathcal{F}}) \cdot D - (s'_\infty)^*(K_{\mathcal{F}}) \cdot D$ is exactly equal to $K_{\mathcal{F}} \cdot B$, the degree of $K_{\mathcal{F}}$ on the bubbles B. By the nefness of $K_{\mathcal{F}}$ this degree is nonnegative, and so

$$(s'_n)^*(K_{\mathcal{F}}) \cdot D \geq (s'_\infty)^*(K_{\mathcal{F}}) \cdot D.$$

We conclude that $(s'_\infty)^*(K_{\mathcal{F}}) \cdot D = 0$ and $F = \emptyset$, that is there are no bubbles at all and the section s_∞ of Lemma 5 is defined over the full $\overline{D}$.

Let us resume. We have proved that *given any sequence of discs* $\{\overline{D_{t_n}} \subset M_{t_n}\}$ *(with* $\partial D_{t_n} \subset S_{t_n}$*),* $t_n \to t_0$*, we may extract a convergent subsequence* $\overline{D_n}$*, whose limit* $\overline{D_\infty}$ *is the image of a section* $s_\infty : \overline{D} \to U_T$*, holomorphic up to the boundary*. It is now a rather standard fact [B-G] [For] that these limits (varying the initial sequence $\{\overline{D_{t_n}}\}$) glue together to a real analytic Levi-flat M_{t_0} with $\partial M_{t_0} = S_{t_0}$, and that $\{M_t\}_{t\in(0,t_0]}$ is a Levi-flat extension of $\{S_t\}_{t\in(0,t_0]}$.

Indeed, observe that if $D_\infty \subset U_T$ is a limit disc with $\partial D_\infty \subset S_{t_0}$ then D_∞ belongs to a (unique) real analytic family of discs $D^\epsilon_\infty \subset U_T$ with $\partial D^\epsilon_\infty \subset S_{t_0}$ ($D^0_\infty = D_\infty$, $\epsilon \in \mathbf{R}$, $|\epsilon|$ small), according to [B-G, §5]: by an easy continuity argument, the "winding number" that appears in [B-G] is zero, because it is zero for $D_n \to D_\infty$. This family D^ϵ_∞ can be deformed to a family $D^\epsilon_n \subset U_T$ with $\partial D^\epsilon_n \subset S_{t_n}$, $D^0_n = D_n \subset M_{t_n}$ (again by [B-G, §5]). This last family must coincide with the family already provided by M_{t_n}, by [B-G] or by the uniqueness property discussed before Lemma 4. Hence the discs D^ϵ_∞, and not only $D^0_\infty = D_\infty$, are contained in the closure of $\cup_{t\in(0,t_0)} M_t$, and they also are limits of discs inside the lower Levi-flats. It is then clear that M_t converges, as $t \to t_0$, to a real analytic Levi-flat M_{t_0} with $\partial M_{t_0} = S_{t_0}$ and with all the other desired properties.

This concludes the proof of the Main Lemma.

4. Hilbert modular foliations and the Monge–Ampère equation

Let Γ be a subgroup of $Aut(\mathbf{D} \times \mathbf{D}) = PSL(2, \mathbf{R}) \times PSL(2, \mathbf{R})$ which acts on $\mathbf{D} \times \mathbf{D}$ in a properly discontinuous way. Suppose that Γ does not contain a finite index subgroup of the type $\Gamma_1 \times \Gamma_2$, with $\Gamma_i \subset PSL(2, \mathbf{R})$. The quotient $\mathbf{D} \times \mathbf{D}/\Gamma$ is a normal complex surface, whose singularities are of Hirzebruch–Jung type. This quotient may be noncompact, but in some cases [Hir] it can be compactified by adding one or more cycles of rational curves ("cusps"), in such a way that the resulting surface is smooth along these cycles. The resulting compact algebraic surface, or even the initial surface $\mathbf{D} \times \mathbf{D}/\Gamma$ when already compact, is called **Hilbert modular surface**. In other words, an Hilbert modular surface is a compact algebraic surface X which contains a (possibly empty) union of disjoint cycles of rational curves D, such that X is smooth along D and $X \setminus D = \mathbf{D} \times \mathbf{D}/\Gamma$, with Γ satisfying the non-split property above (this terminology is not the standard one, because we do not require that Γ has an arithmetic origin, but it is the most suitable for our purposes).

Clearly, an Hilbert modular surface has two natural foliations, arising from the vertical and the horizontal ones on $\mathbf{D} \times \mathbf{D}$ and called **Hilbert modular foliations**. These two foliations are tangent to the cycles of rational curves, and transverse each other outside these cycles. These rational curves are the only algebraic leaves of the two foliations (this follows from the non-split property of Γ).

One can quite easily check that any Hilbert modular foliation is nef, its singularities are reduced (in the sense of Seidenberg) and its Kodaira dimension $kod(\mathcal{F})$ is $-\infty$, i.e. $h^0(X, K_{\mathcal{F}}^{\otimes n}) = 0$ for every positive n. We refer to [McQ] and [Bru] for these facts, and other basic facts concerning the Kodaira dimension of foliations. The main question left open in [McQ] and [Bru] is the following

Conjecture. Let $\mathcal{F}$ be a nef foliation with reduced singularities on a compact complex algebraic surface X. Suppose that $kod(\mathcal{F}) = -\infty$. Then $\mathcal{F}$ is a Hilbert modular foliation.

Let $\mathcal{F}$ satisfy the hypotheses of the Conjecture. The following properties can be extracted from [McQ] and [Bru] (they are all easy to prove, except the last one):

i) $H^1(X, \mathcal{O}_X) = 0$
ii) $c_1(K_{\mathcal{F}}) \in H^2(X, \mathbf{R})$ is not trivial
iii) $c_1^2(K_{\mathcal{F}}) = 0$
iv) $\mathcal{F}$ has a finite number of algebraic leaves
v) $\mathcal{F}$ has a finite number of parabolic leaves, and they are all algebraic.

Remark that Hilbert modular foliations always appear in pairs, horizontal and vertical. Thus, in trying to prove the Conjecture it is quite natural to look for the "companion foliation" of $\mathcal{F}$. The following program has been proposed in [McQ] and [Bru]. Take a (singular) hermitian metric on $K_{\mathcal{F}}$ whose curvature Ω is a closed positive current [Dem]. Suppose that we are able to define the wedge product $\Omega \wedge \Omega$, as a closed positive current representing $c_1^2(K_{\mathcal{F}})$. Then, by iii), we obtain the (homogeneous) **Monge–Ampère equation** $\Omega \wedge \Omega \equiv 0$, and we may hope that this equation generates a **Monge–Ampère foliation** $\mathcal{G}$. See for instance [Kli] for the basic theory of the Monge–Ampère operator. Optimistically, we may also hope that the foliation $\mathcal{G}$ is *holomorphic*, even if this is a quite exceptional fact:

Monge–Ampère foliations have complex leaves but in general they are not holomorphic. If it is the case, then the Conjecture follows rather quickly:

Proposition 4 [Bru] [McQ]. *Let $\mathcal{F}$ be a nef foliation with reduced singularities and with Kodaira dimension $-\infty$. Let Ω be the positive curvature of a singular hermitian metric on $K_{\mathcal{F}}$, and suppose that there exists a holomorphic foliation $\mathcal{G}$ in the kernel of Ω (more precisely, this means that if ω is a local holomorphic 1-form definining $\mathcal{G}$ then $\omega \wedge \Omega \equiv 0$). Then $\mathcal{F}$ is a Hilbert modular foliation.*

This is proved in the works above with the help of a deep theorem of Yau and Tian [Tia] on Kähler–Einstein metrics, and without any hypothesis on the singular metric on $K_{\mathcal{F}}$ (except the positivity of its curvature, of course). Let us henceforth specialize to the case where Ω is the curvature of the *canonical Poincaré metric*, which is certainly nontrivial thanks to the property v) recalled above and which is positive by our Theorem. We shall see below that, in that case, the previous proposition can be proved in a relatively elementary way. Moreover, and much more important, we shall see that the hypothetical foliation $\mathcal{G}$ indeed exists, leading to a full proof of the Conjecture above.

Recall (McQuillan's Proposition 3) that the canonical Poincaré metric is continuous outside the algebraic leaves and the singularities of the foliation. Let us now state another regularity result, concerning however its curvature Ω rather than the metric itself. It is mainly a corollary to Demailly's work [Dem], and it allows (at least) to give a well-defined sense to the Monge–Ampère equation $\Omega \wedge \Omega \equiv 0$.

Proposition 5. *Let $\mathcal{F}$ be a nef foliation with reduced singularities and with Kodaira dimension $-\infty$. Let Ω be the curvature of the canonical Poincaré metric. Then Ω has everywhere vanishing Lelong number, it is absolutely continuous (i.e. it is a $(1,1)$-form with L^1_{loc}-coefficients) and it satisfies the equation $\Omega \wedge \Omega \equiv 0$, in a punctual sense.*

Proof. Let us firstly consider the Siu decomposition of Ω [Dem, 2.18]:

$$\Omega = \sum \lambda_j \delta_{C_j} + \Omega_{res} = \Omega_{alg} + \Omega_{res}$$

with $\lambda_j > 0$, δ_{C_j} = integration current over the algebraic curve $C_j \subset X$, Ω_{res} = closed positive current with vanishing Lelong number outside a countable set. Each C_j is clearly the closure of a parabolic leaf, so that the sum is finite thanks to iv) above. From $[\Omega]$ nef and $[\Omega]^2 = 0$ it follows that $[\Omega] \cdot C_j = 0$ for every j and $[\Omega] \cdot [\Omega_{res}] = 0$. Moreover, $[\Omega_{res}]$ is also nef, so that by Hodge index theorem [BPV] we deduce that $[\Omega_{res}]$ is proportional to $[\Omega]$ (which is not zero by ii) above). Hence $[\Omega_{alg}]$ also is proportional to $[\Omega]$: for some $\lambda \geq 0$ we have $[\Omega_{alg}] = \lambda \cdot [\Omega]$. From $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, K_{\mathcal{F}}^{\otimes n}) = 0$ for every $n \geq 1$ it follows that $[\Omega]$ cannot be represented by an effective divisor, hence $\lambda = 0$, that is $\Omega_{alg} = 0$ and $\Omega = \Omega_{res}$. Finally, from $[\Omega]^2 = 0$, $\Omega = \Omega_{res}$ and [Dem, 9.6] it follows that the Lelong number of Ω is zero *everywhere*, and not only outside a countable subset.

This last property allows to use the approximation theorem [Dem, 9.1] in its simplest form: there exists a sequence of smooth closed (1,1)-forms Ω_k on X, $k \in \mathbf{N}$, such that:

1) $[\Omega_k] = [\Omega]$ for every k;
2) $\Omega_k \geq -\frac{1}{k}\omega$ for every k, where ω is a fixed Kähler form on X;
3) $\Omega_k \xrightarrow{weak} \Omega$ as $k \to +\infty$.

From 2) and $\int_X \Omega_k \wedge \Omega = [\Omega]^2 = 0$ for every k it follows also:

4) $\Omega_k \wedge \Omega \xrightarrow{weak} 0$ as $k \to +\infty$.

Take a local chart $U \subset X''$ disjoint from the algebraic leaves and with coordinates (z, w) such that $\mathcal{F} = \{dz = 0\}$. We therefore have a psh function F on U such that

$$\Omega = \frac{i}{2\pi}\partial\bar{\partial}F \qquad \text{and} \qquad F_{w\bar{w}} = e^F$$

and moreover F is continuous by Proposition 3 (in the following we shall need only the local boundedness of F). By construction [Dem, §9], the regularizing forms Ω_k can be choosen in such a way that, on U, we have $\Omega_k = \frac{i}{2\pi}\partial\bar{\partial}G_k$ and $G_k \searrow F$. Therefore, for every $\varphi \in C_c^\infty(U)$ and by 4) above:

$$\int_U F\cdot(i\partial\bar{\partial}\varphi)\wedge\Omega = \lim_{k\to+\infty}\int_U G_k\cdot(i\partial\bar{\partial}\varphi)\wedge\Omega = \lim_{k\to+\infty} 2\pi\int_U \varphi\cdot\Omega_k\wedge\Omega = 0$$

(this is simply the weak form of the Monge–Ampère equation $\Omega \wedge \Omega \equiv 0$).

Let $\Omega = \Omega_{ac} + \Omega_{sg}$ be the Lebesgue decomposition of Ω into absolutely continuous and singular parts. In the chart U, the equation $F_{w\bar{w}} = e^F \in L^\infty_{loc} \subset L^1_{loc}$ shows that the coefficient of $idw \wedge d\bar{w}$ in Ω_{sg} is identically zero, and therefore the same holds for the coefficients of $idz \wedge d\bar{w}$ and $idw \wedge d\bar{z}$ (for $\Omega_{sg} \geq 0$). Thus $F_{z\bar{w}}$ and $F_{w\bar{z}}$ belong to L^1_{loc}, and

$$\Omega_{ac} = Midz \wedge d\bar{z} + e^F idw \wedge d\bar{w} + F_{z\bar{w}}idz \wedge d\bar{w} + F_{w\bar{z}}idw \wedge d\bar{z}$$

$$\Omega_{sg} = midz \wedge d\bar{z}$$

with $F_{z\bar{z}} = m + M$, $M \ll$ Lebesgue, $m \perp$ Lebesgue.

Still working on U, we may regularize F by taking the convolution with regularizing kernels ρ_k depending only on the variable z (because F is already smooth in w !):

$$F_k(z, w) = \int_{\mathbf{C}} F(t, w)\rho_k(z - t)idt \wedge d\bar{t}.$$

These functions are smooth, psh, and $F_k \searrow F$. For every $\varphi \in C_c^\infty(U)$ we thus have

$$\int_U \varphi \cdot (i\partial\bar{\partial}F_k) \wedge \Omega = \int_U F_k \cdot (i\partial\bar{\partial}\varphi) \wedge \Omega \overset{k\to+\infty}{\longrightarrow} \int_U F \cdot (i\partial\bar{\partial}\varphi) \wedge \Omega = 0$$

that is $(i\partial\bar{\partial}F_k) \wedge \Omega \overset{weak}{\longrightarrow} 0$ as $k \to +\infty$. Since $i\partial\bar{\partial}F_k \geq 0$, this implies $(i\partial\bar{\partial}F_k) \wedge \Omega_{sg} \overset{weak}{\longrightarrow} 0$ as $k \to +\infty$, that is, due to the particular local structure of Ω_{sg},

$$(F_k)_{w\bar{w}} \cdot m \overset{weak}{\longrightarrow} 0 \qquad \text{as} \qquad k \to +\infty.$$

Observe now that the convexity of the exponential function and the identity $F_{w\bar{w}} = e^F$ give $(F_k)_{w\bar{w}} \geq e^{F_k}$, and consequently

$$e^{F_k} \cdot m \overset{weak}{\longrightarrow} 0 \qquad \text{as} \qquad k \to +\infty.$$

But $e^{F_k} \cdot m$ also converge to $e^F \cdot m$, so that this last measure must be identically zero, as well as m because F is locally bounded. Thus $\Omega_{sg}|_U \equiv 0$.

In conclusion, we have proved that *Supp* (Ω_{sg}) is contained in the union of the algebraic leaves and the singularities of the foliation. From the vanishing of the Lelong numbers of Ω it follows that the Lelong numbers of Ω_{sg} are well defined and equal to zero everywhere. From Lebesgue's density theorem we then deduce that Ω_{sg} is identically zero, and so Ω is absolutely continuous.

Finally, the vanishing of the pointwise product $\Omega \wedge \Omega$ follows again from $[\Omega]^2 = 0$ and [Dem, 9.5]. □

The most remarkable property of the canonical Poincaré metric, in the case $kod(\mathcal{F}) = -\infty$, is however expressed by the following local result.

Proposition 6. *Let F be a locally bounded psh function on $\mathbf{D} \times \mathbf{D}$ such that:*

1) $\Omega = \frac{i}{2\pi}\partial\bar{\partial}F$ *is absolutely continuous;*
2) $\Omega \wedge \Omega \equiv 0$*;*
3) $F_{w\bar{w}} = e^F$.

Then there exists a holomorphic foliation $\mathcal{G}$ on $\mathbf{D} \times \mathbf{D}$ in the kernel of Ω and transverse to the verticals.

Proof. The equation $\Omega \wedge \Omega \equiv 0$ can be rewritten as

$$F_{z\bar{z}} = F_{z\bar{w}} F_{w\bar{z}} e^{-F}$$

and the kernel of Ω is given by

$$\frac{dw}{dz} = -F_{z\bar{w}} e^{-F}$$

hence our aim is to prove that the function $G = F_{z\bar{w}}e^{-F}$ is *holomorphic*. This is just a direct computation, but some care is necessary due to the (a priori) non-smoothness of F.

Let us firstly observe that from $F, F_{z\bar{z}} \in L^1_{loc}$ it follows that also F_z and $F_{\bar{z}}$ belong to L^1_{loc}. Indeed, by Poisson formula in the z-variable we have

$$F(z,w) = \frac{1}{2\pi}\int_{\partial \mathbf{D}} F(e^{i\theta}, w)\phi(\theta; z)d\theta + \int_{\mathbf{D}} F_{z\bar{z}}(t,w)g(t;z)idt \wedge d\bar{t}$$

where $\phi(\theta; z)d\theta$ is the harmonic measure on $\partial\mathbf{D}$ seen from z and $g(t; z)$ is the Green function on $\mathbf{D}$ with pole at z. The first integral is smooth in z and its z-derivative is L^1_{loc} because F is. The z-derivative of the second integral is also L^1_{loc}, because $\frac{\partial g}{\partial z} = \frac{1}{z-t} + \{\text{smooth function}\}$ is locally integrable, and the convolution of two locally integrable functions is still locally integrable. Similarly, from $F_{w\bar{w}} \in L^1_{loc}$ we deduce $F_w, F_{\bar{w}} \in L^1_{loc}$, and therefore $F \in W^{1,1}_{loc}$.

Because $exp : \mathbf{R} \to \mathbf{R}$ has bounded derivative on the range of F, the function e^F is also in $W^{1,1}_{loc}$ and moreover $(e^F)_z = e^F F_z$, $(e^F)_{\bar{z}} = e^F F_{\bar{z}}$, etc. (see, for instance, H. Brezis, Analyse Fonctionnelle, Prop. IX.5). As a distribution, F_z satisfies the differential equation $(F_z)_{w\bar{w}} = (F_{w\bar{w}})_z = (e^F)_z = e^F F_z$, and elliptic regularity (in the vertical direction) gives: $F_z(z,\cdot) \in C^\infty(\mathbf{D})$ for almost every $z \in \mathbf{D}$. The same for $F_{\bar{z}}$. We can therefore compute the classical derivative $\frac{\partial F_{\bar{z}}}{\partial w}$, which is of course equal (a.e.) to the distributional derivative $F_{\bar{z}w}$ because this last one is L^1_{loc}. Hence also $F_{w\bar{z}}$ and $F_{z\bar{w}}$ are smooth on almost every vertical, as well as $F_{z\bar{z}} = F_{z\bar{w}}F_{w\bar{z}}e^{-F}$.

Now we can compute the classical derivative $\frac{\partial^2 F_{z\bar{z}}}{\partial w \partial \bar{w}}$. We can apply Leibniz rule to the product $F_{z\bar{w}}F_{w\bar{z}}e^{-F}$, each factor being smooth on a.e. vertical. Using $F_{w\bar{w}} = e^F$ and replacing classical derivatives with distributional ones whenever possible (i.e., whenever these last ones are L^1_{loc}) we find:

$$\frac{\partial^2 F_{z\bar{z}}}{\partial w \partial \bar{w}} = (F_{z\bar{z}} + F_z F_{\bar{z}})e^F + \left|\frac{\partial F_{z\bar{w}}}{\partial \bar{w}} - F_{z\bar{w}}F_{\bar{w}}\right|^2 e^{-F}.$$

This function is measurable, smooth on almost every vertical, and *positive*. It then follows that it is L^1_{loc}: if $\phi \in C^\infty_c(\mathbf{D} \times \mathbf{D})$ then $z \mapsto \int_{\mathbf{D}} |\frac{\partial^2 F_{z\bar{z}}}{\partial w \partial \bar{w}}(z,w)| \cdot \phi(z,w)idw \wedge d\bar{w} = \int_{\mathbf{D}} F_{z\bar{z}}(z,w)\phi_{w\bar{w}}(z,w)idw \wedge d\bar{w}$ belongs to $L^1(\mathbf{D})$ (for $F_{z\bar{z}}$ belongs to $L^1_{loc}(\mathbf{D} \times \mathbf{D})$) and hence $\frac{\partial^2 F_{z\bar{z}}}{\partial w \partial \bar{w}}$ belongs to $L^1_{loc}(\mathbf{D} \times \mathbf{D})$ by Tonelli theorem. This local integrability implies that $\frac{\partial^2 F_{z\bar{z}}}{\partial w \partial \bar{w}}$ is in fact equal to the distributional derivative $(F_{z\bar{z}})_{w\bar{w}}$. Moreover, we also have $(F_{z\bar{z}} + F_z F_{\bar{z}})e^F \in L^1_{loc}$ (and consequently $F_z, F_{\bar{z}} \in L^2_{loc}$, for $F \in L^\infty_{loc}$) and $|\frac{\partial F_{z\bar{w}}}{\partial \bar{w}} - F_{z\bar{w}}F_{\bar{w}}|^2 e^{-F} \in L^1_{loc}$.

On the other side, let us compute $(F_{w\bar{w}})_{z\bar{z}} = (e^F)_{z\bar{z}}$. Because $F_{z\bar{z}} \in L^1_{loc}$ and $F_z, F_{\bar{z}} \in L^2_{loc}$, we have $(e^F)_{z\bar{z}} \in L^1_{loc}$ and

$$(e^F)_{z\bar{z}} = (F_{z\bar{z}} + F_z F_{\bar{z}})e^F$$

(for instance, by the same arguments of H. Brezis, loc. cit., Prop. IX.5).

By comparing $(F_{z\bar{z}})_{w\bar{w}} = \frac{\partial^2 F_{z\bar{z}}}{\partial w \partial \bar{w}}$ and $(F_{w\bar{w}})_{z\bar{z}} = (e^F)_{z\bar{z}}$ we finally obtain

$$\left| \frac{\partial F_{z\bar{w}}}{\partial \bar{w}} - F_{z\bar{w}} F_{\bar{w}} \right|^2 e^{-F} = 0 \qquad a.e.$$

that is

$$\frac{\partial F_{z\bar{w}}}{\partial \bar{w}} = F_{z\bar{w}} F_{\bar{w}} \qquad a.e.$$

Consider now $G = F_{z\bar{w}} e^{-F}$, which is L^1_{loc} and smooth on almost every vertical. We have $\frac{\partial G}{\partial \bar{w}} = (\frac{\partial F_{z\bar{w}}}{\partial \bar{w}} - F_{z\bar{w}} \frac{\partial F}{\partial \bar{w}})e^{-F} = 0$ a.e. by the previous identity, hence G is holomorphic on almost every vertical and also its distributional derivative $G_{\bar{w}}$ is 0.

The computation of $G_{\bar{z}}$ is a little more elaborated. Using $\frac{\partial F_{z\bar{w}}}{\partial \bar{w}} = F_{z\bar{w}} F_{\bar{w}}$ we find $\frac{\partial F_{z\bar{z}}}{\partial \bar{w}} = F_{\bar{z}} F_{z\bar{w}}$, which is L^1_{loc} because both factors are L^2_{loc}. Hence the distributional derivative $(F_{z\bar{w}})_{\bar{z}} = (F_{z\bar{z}})_{\bar{w}}$ is also L^1_{loc} and equal to $F_{\bar{z}} F_{z\bar{w}}$. We also have $(e^{-F})_{\bar{z}} = -(e^{-F})F_{\bar{z}} \in L^2_{loc}$, for $F \in L^\infty_{loc}$ and $F_{\bar{z}} \in L^2_{loc}$. Therefore $(F_{z\bar{w}})_{\bar{z}} e^{-F} \in L^1_{loc}$, $F_{z\bar{w}}(e^{-F})_{\bar{z}} \in L^1_{loc}$, and the following Leibniz formula holds:

$$G_{\bar{z}} = \left(F_{z\bar{w}} e^{-F}\right)_{\bar{z}} = (F_{z\bar{w}})_{\bar{z}} e^{-F} + F_{z\bar{w}}(e^{-F})_{\bar{z}}$$

(see, for instance, H. Brezis, loc. cit., Prop. IX.4, mutatis mutandis). Hence $G_{\bar{z}} = F_{\bar{z}} F_{z\bar{w}} e^{-F} - F_{z\bar{w}} e^{-F} F_{\bar{z}} = 0$, and G is holomorphic on $\mathbf{D} \times \mathbf{D}$. □

After this preparation, we can now prove the Conjecture stated at the beginning of this section.

Corollary. *Let $\mathcal{F}$ be a nef foliation with reduced singularities and with Kodaira dimension $-\infty$. Then $\mathcal{F}$ is a Hilbert modular foliation.*

Proof. By Propositions 3, 5 and 6 we certainly have a holomorphic foliation $\mathcal{G}$ in the kernel of Ω (and transverse to $\mathcal{F}$) outside the algebraic leaves and the singularities of $\mathcal{F}$, and by Proposition 4 we only need to check that this foliation $\mathcal{G}$ extends to the full X. Of course, it is sufficient to extend to X'', whose complement in X has codimension 2.

Let $U \subset X''$ be a local chart with coordinates (z, w) such that $\mathcal{F} = \{dz = 0\}$. Let F be the local weight of the canonical Poincaré metric, so that the kernel of $\Omega = \frac{i}{2\pi} \partial \bar{\partial} F$ is expressed by (cfr. proof of Proposition 6)

$$\frac{dw}{dz} = -F_{z\bar{w}} e^{-F} = -G.$$

Note that $F_{z\bar{w}} \in L^2_{loc}$ (for $|F_{z\bar{w}}|^2 = F_{z\bar{z}}e^F \in L^1_{loc}$) and $e^{-F} \in L^2_{loc}$ (zero Lelong numbers), hence $G \in L^1_{loc}$. Moreover, we already know that G is holomorphic outside $\Gamma = U \cap \{\text{algebraic leaves}\}$, which is a finite union of verticals in U. We obtain that G is in fact meromorphic on U, with (at most) first order poles along Γ (by Cauchy formula, $f \in L^1_{loc}(\mathbf{D}) \cap \mathcal{O}(\mathbf{D}^*)$ implies $zf(z) \in \mathcal{O}(\mathbf{D})$). This means that $\mathcal{G}$ holomorphically extends to the full U. □

We conclude with the promised proof of Proposition 4.

Proof of Proposition 4 (for the Poincaré metric). The beginning is as in [Bru, pp. 134–135], but let us give anyway some details (and some simplifications) for reader's convenience. We assume however some familiarity with the most basic techniques of [Bru] or [McQ].

Let D be the tangency divisor between $\mathcal{F}$ and $\mathcal{G}$. By construction of $\mathcal{G}$, it is composed by $\mathcal{F}$-invariant algebraic curves, which therefore are also $\mathcal{G}$-invariant. Moreover, the extension argument of the previous proof ($G \in L^1_{loc}$) shows that $\mathcal{F}$ and $\mathcal{G}$ have only first order tangencies, that is D is reduced. Remark also that D is a normal crossing divisor, for $Sing(\mathcal{F})$ are reduced; we may suppose (up to a blow-up which does not affect the data) that each irreducible component of D is smooth. Note that D obviously contains all the singularities of $\mathcal{F}$ and $\mathcal{G}$.

Let D_j be an irreducible component of D. Recall the formula $N_{\mathcal{G}} \cdot D_j = D_j^2 + Z(\mathcal{G}, D_j)$, where $Z(\mathcal{G}, D_j)$ is the number of singularities of $\mathcal{G}$ along D_j (if $\mathcal{G}$ has a singularity at a point of $Sing(X) \cap D_j$, then that singularity must be counted in the orbifold sense, i.e. by computing the multiplicity in the local smooth covering and then by dividing the result by the order of the covering).

We have $\mathcal{O}_X(D) = K_{\mathcal{F}} \otimes N_{\mathcal{G}}$ and so

$$D \cdot D_j = K_{\mathcal{F}} \cdot D_j + D_j^2 + Z(\mathcal{G}, D_j).$$

But clearly $Z(\mathcal{G}, D_j) \geq (\cup_{k \neq j} D_k) \cdot D_j$, and $K_{\mathcal{F}} \cdot D_j \geq 0$ by nefness, so that we must have $K_{\mathcal{F}} \cdot D_j = 0$ and $Z(\mathcal{G}, D_j) = (\cup_{k \neq j} D_k) \cdot D_j$, for every j. This last fact means that $\mathcal{G}$ along D_j is singular only at the intersection points of D_j with the other components D_k, $k \neq j$, and moreover each singularity is nondegenerate (both eigenvalues are not zero). We shall write $Sing(\mathcal{G}) = Sing(D)$ to synthetize such a property. In particular, around each $p \in Sing(X)$ the foliation $\mathcal{G}$ has the good quotient type structure, and $Sing(\mathcal{G}) \cap Sing(X) = \emptyset$. Moreover, a simple local computation, based on the fact that $\mathcal{F}$ and $\mathcal{G}$ have only a first order tangency along D, shows that $Sing(X) \cap D = \emptyset$.

From $K_{\mathcal{F}} \cdot D_j = 0$ for every j we deduce that the intersection form on $\{D_j\}$ is negative definite (otherwise $kod(\mathcal{F}) \geq 0$, by Hodge). Then from $K_{\mathcal{F}} \cdot D_j = -\chi(D_j) + Z(\mathcal{F}, D_j)$ we easily find that each connected component D' of D is a chain or a cycle of rational curves; moreover, in the latter case $\mathcal{F}$ along D' is singular only at the crossing points and these singularities are nondegenerate. In fact, no component of D is a chain: from $Sing(\mathcal{G}) = Sing(D)$ and Camacho-Sad formula [Bru] we would find a contradiction with the negativity of the intersection form. As a first conclusion we thus obtain that around D the two foliations have the Hilbert modular shape: each connected component of D is a contractible cycle of rational curves and

$$Sing(\mathcal{F}) = Sing(D) = Sing(\mathcal{G}).$$

We can also compute the eigenvalues of the singularities of $\mathcal{G}$, still using Camacho-Sad formula, and we find that all these singularities are reduced. Moreover $K_{\mathcal{G}}$ is nef: otherwise there would exists a $\mathcal{G}$-invariant rational curve over which $K_{\mathcal{G}}$ has negative degree, a possibility which is easily excluded by $Sing(\mathcal{G}) = Sing(D)$.

Note that

$$K_{\mathcal{G}} = N^*_{\mathcal{F}} \otimes \mathcal{O}_X(D),$$

the logarithmic conormal bundle of $\mathcal{F}$. We deduce from this that $K_{\mathcal{F}} \cdot K_{\mathcal{G}} = K_{\mathcal{F}} \cdot N^*_{\mathcal{F}} > 0$, the last inequality resulting from $kod(\mathcal{F}) = -\infty$ and Riemann-Roch (working on the minimal smooth resolution $\tilde{X}$ of X, where Riemann-Roch can be safely applied to integral multiples of the pull-back L of $K_{\mathcal{F}}$: we have $L \cdot L = 0$ and therefore $L \cdot K_{\tilde{X}} > 0$, but $L \cdot K_{\tilde{X}} = L \cdot K_{\tilde{\mathcal{F}}} + L \cdot N^*_{\tilde{\mathcal{F}}} = K_{\mathcal{F}} \cdot N^*_{\mathcal{F}}$). In particular, $K_{\mathcal{G}}$ is not numerically trivial. The logarithmic Castelnuovo - De Franchis - Bogomolov lemma says that $kod(K_{\mathcal{G}}) \leq 0$, for $\mathcal{F}$ is not a fibration, but the case $kod(K_{\mathcal{G}}) = 0$ is excluded because in that case $K_{\mathcal{G}}$ would be a torsion line bundle [McQ] [Bru] and therefore numerically trivial.

Hence $\mathcal{G}$ is a nef foliation with reduced singularities and with Kodaira dimension $-\infty$. We may apply to $\mathcal{G}$ the previous Propositions 3, 5 and 6 and the extension argument of the proof of the previous Corollary, in order to construct a holomorphic Monge–Ampère foliation $\mathcal{F}'$ in the kernel of the curvature of the Poincaré metric on $\mathcal{G}$. Of course, the tangency divisor between $\mathcal{G}$ and $\mathcal{F}'$ is the same D as before, being reduced and composed by (all the) cycles of $\mathcal{G}$-invariant rational curves. Therefore $K_{\mathcal{F}'} = K_{\mathcal{F}}$, both bundles being equal to $N^*_{\mathcal{G}} \otimes \mathcal{O}_X(D)$. It follows from this that $\mathcal{F}' = \mathcal{F}$: otherwise the tangency divisor between $\mathcal{F}'$ and $\mathcal{F}$ would be of the form $D + E$, $E \geq 0$, and so $K_{\mathcal{F}}^{\otimes 2} = K_{\mathcal{F}} \otimes K_{\mathcal{F}'} = K_X \otimes \mathcal{O}_X(D + E)$ would be big because $K_X \otimes \mathcal{O}_X(D) = K_{\mathcal{F}} \otimes K_{\mathcal{G}}$ is big (for $K_{\mathcal{F}} \cdot K_{\mathcal{G}} > 0$).

Hence on $X \setminus D$ we have two nonsingular transverse foliations $\mathcal{F}_0 = \mathcal{F}|_{X \setminus D}$ and $\mathcal{G}_0 = \mathcal{G}|_{X \setminus D}$, all of whose leaves are hyperbolic, and such that $\mathcal{G}_0$ (resp. $\mathcal{F}_0$) induces, by its holonomy, local isometries between the leaves of $\mathcal{F}_0$ (resp. $\mathcal{G}_0$). Indeed, $\mathcal{G}_0 = Ker\ \Omega$ and $\Omega|_{\mathcal{F}_0}$ = hyperbolic area form, and any local biholomorphism between hyperbolic curves which preserves the hyperbolic area forms is in fact a hyperbolic isometry. In other words, $X \setminus D$ admits a $PSL(2, \mathbf{R}) \times PSL(2, \mathbf{R})$-structure (i.e., an atlas with values in $\mathbf{D} \times \mathbf{D}$ and with coordinate changes in $Aut(\mathbf{D} \times \mathbf{D})$), and such a structure is complete because of the completeness of the Poincaré metrics along $\mathcal{F}_0$ and $\mathcal{G}_0$. It is then a standard differential-geometric fact that $X \setminus D$ is a quotient of $\mathbf{D} \times \mathbf{D}$, and (by construction) $\mathcal{F}_0$ and $\mathcal{G}_0$ lift to the horizontal and vertical foliations on $\mathbf{D} \times \mathbf{D}$. Therefore $\mathcal{F}$, as well as $\mathcal{G}$, is Hilbert modular. □

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