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On Kähler surfaces with semipositive Ricci curvature

Abstract. We study the existence problem for smooth or real analytic Kähler metrics with semipositive Ricci curvature on the complex projective plane blown up at 9 points.

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We shall study in this note a problem from [**7**].

Take a smooth elliptic curve $C_0 \subset \mathbb{C}P^2$ and blow-up 9 points $p_1, \ldots, p_9$ on C_0. The result is a compact complex surface S which contains a smooth elliptic curve C with zero selfintersection, the strict transform of C_0. The anticanonical bundle K_S^{-1} of S is isomorphic to $\mathcal{O}_S(C)$, and it is *nef*, i.e. it has nonnegative degree on every compact complex curve in S. Demailly, Peternell and Schneider ask in [**7**] about the existence of a smooth hermitian metric on K_S^{-1} whose curvature is a semipositive $(1,1)$-form. According to Aubin-Yau Theorem (solution of the Calabi conjecture), this is equivalent to the existence of a smooth Kähler metric on S whose Ricci curvature is semipositive.

A simple case is when the normal bundle of C in S, $N_C \in \mathrm{Pic}^\circ(C)$, is torsion, say of order ℓ. Then a standard argument shows that S admits an elliptic fibration $\pi : S \to \mathbb{C}P^1$, such that $\pi^{-1}(\infty) = \ell C$. Hence $(K_S^{-1})^{\otimes \ell}$ is isomorphic to $\pi^*(\mathcal{O}(1))$, and it is immediate to find (by pull-back) the desired smooth metric on K_S^{-1} with semipositive curvature.

Another simple case is when N_C satisfies a certain diophantine condition introduced in [**15**, p. 595] (see also [**1**]), which is opposed to torsionness and almost

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surely true. Then, by [**15**, Th. 3], we get on some neighborhood U of C a function $F_0 : U \to \mathbb{R} \cup \{+\infty\}$ which is pluriharmonic outside of C and has –logarithmic poles along C, i.e.

$$\mathrm{dd}^c F_0 = -\delta_C.$$

One says that C has a *pseudoflat* neighborhood in S, which is foliated by the Levi-flat 3-tori $\{F_0 = c\}$, with c large. By an easy regularization procedure, we then find a smooth plurisubharmonic function $F : S \setminus C \to \mathbb{R}$, which coincides with F_0 on some smaller neighborhood V of C: just take $F = \varphi \circ F_0$ for a suitably chosen convex function $\varphi \in C^\infty(\mathbb{R})$, equal to 0 on a neighborhood of $-\infty$ and to *id* on a neighborhood of $+\infty$. Then F defines a smooth metric on $K_S^{-1} = \mathcal{O}_S(C)$ (choose a reference section s with $\{s = 0\} = C$ and set $\|s\| = \exp(-F)$), and the curvature of this metric is semipositive.

Actually, to find a metric on K_S^{-1} with semipositive curvature is *equivalent* to find on $S \setminus C$ a smooth plurisubharmonic function which growths like $-\log \mathrm{dist}(C, \cdot)$ (up to a function which smoothly extends to C).

There is however a substantial difference between the two examples above. In the first case we can get a metric whose curvature has full support in S, that is we have a positive eigenvalue on an open and dense subset of S. In the second case, instead, we typically get a metric whose curvature is concentrated in the (possibly small) neighborhood U, since the evoked regularization procedure produces a function F which is constant outside of U. For physical reasons, it would be preferable to find a metric on K_S^{-1} whose curvature is spread over the full S, a sort of heat-equation regularization of δ_C. A natural condition which ensures such a property is the *real analyticity* of the metric. However, we shall prove that real analytic metrics with semipositive curvature rarely exist:

Theorem 1. *Suppose that $S \setminus C$ does not contain any compact complex curve. Then:*

(i) *There exists on S a smooth Kähler metric with semipositive Ricci curvature if and only if C admits a pseudoflat neighborhood in S.*

(ii) *There does not exist on S a real analytic Kähler metric with semipositive Ricci curvature.*

As is well known, the condition imposed on $S \setminus C$ is very generic in our parameter space C_0^9, to which $(p_1, \ldots, p_9)$ belongs (see e.g. [**11**, (4.14)]). Observe also that, by the Hodge Index Theorem, the adjunction formula and $K_S = \mathcal{O}_S(-C)$, any irreducible compact curve D in $S \setminus C$ is either a (-2)-curve (smooth, rational, with selfintersection -2) or an arithmetically elliptic curve with zero selfintersection. In the latter case, however, D is necessarily cohomologous to αC, for some rational $\alpha > 0$, and S has an elliptic fibration containing C and D as fibers, so that K_S^{-1} clearly admits a real analytic metric with semipositive curvature as explained before. Hence the only case which remains to be analysed is when $S \setminus C$ contains some (-2)-curves but no elliptic curves. Our proof will show that even in that case the existence of real analytic metrics with semipositive curvature is exceptional.

Concerning the smooth case, we stress that our result does not yet answer definitely to the question of [**7**]: we do not know if suitable choices of the nine points may produce a curve C *without* pseudoflat neighborhoods. It is likely, however, that those choices exist (compare with [**15**, p. 606]). Another still open problem is about the existence of smooth metrics whose curvature has full support; as we shall see, this is related to finding a pseudoflat neighborhood of C which is dense in S.

The proof of Theorem 1 is quite simple, but in spite of its simplicity we think that it may be interesting to know that for a full measure choice of $(p_1, \dots, p_9)$ we get a surface S which admits a smooth Kähler metric with semipositive curvature, but not a real analytic one.

In this paper "smooth" means "infinitely smooth". We leave to the reader the care to check the minimal degree of differentiability needed for our arguments.

Before starting the proof of Theorem 1, let us state and prove a probably well-known fact concerning weakly 1-complete surfaces, i.e. surfaces which admit a smooth plurisubharmonic exhaustion. It is basically contained in [**12**], up to replacing Nakano's theorem used there with the more powerful Hormander's estimates [**6**].

Proposition 2. *Let W be a connected Kähler surface with trivial canonical bundle, and suppose that there exists a smooth plurisubharmonic exhaustion $f : W \to \mathbb{R}$ which is strictly plurisubharmonic at some point. Then W is holomorphically convex* (*and in particular it is Stein if it does not contain any compact complex curve*).

Proof. By [**12**, Prop. 1.4], it is sufficient to show that there exists a non-constant holomorphic function on W. We shall use for that purpose the argument of [**12**, p. 159].

Take two points $p, q \in W$ around which f is strictly plurisubharmonic. Blow-up them, and let $E, F \subset \widetilde{W}$ be the corresponding exceptional divisors. Set

$$L = \mathcal{O}_{\widetilde{W}}(-2E - 2F).$$

Since L has positive degree on E and on F, it is easy to construct (by using the pull-back of f to $\widetilde{W}$) a metric on L whose curvature is everywhere semipositive, and strictly positive on some neighbourhood V of $E \cup F$. Remark also that, since K_W is trivial, we have $K_{\widetilde{W}} \otimes L = \mathcal{O}_{\widetilde{W}}(-E-F)$, and hence we may take a meromorphic section h of $K_{\widetilde{W}} \otimes L$ which has first order poles along $E \cup F$ and which is holomorphic and nowhere vanishing outside.

Take now $\varphi \in C^{\infty}_{\rm cpt}(\widetilde{W})$ such that $\mathrm{Supp}(\varphi) \subset V$, $\varphi \equiv 0$ around E, $\varphi \equiv 1$ around F. Then $(\overline{\partial}\varphi) \cdot h \in \mathrm{A}^{2,1}(\widetilde{W}) \otimes L$ is $\overline{\partial}$-closed and satisfies the hypotheses of [**6**, Th. 14.2]. Hence, by that theorem, there exists $\vartheta \in \mathrm{A}^{2,0}(\widetilde{W}) \otimes L$ such that $\overline{\partial}\vartheta = (\overline{\partial}\varphi) \cdot h$. The function $\dfrac{\vartheta}{h}$ is smooth on $\widetilde{W}$ and vanishes on $E \cup F$. Thus the function $\varphi - \dfrac{\vartheta}{h}$ is holomorphic, equal to 0 on E and to 1 on F. It projects to W to a nonconstant holomorphic function. $\square$

It is an old problem to know if a statement like the previous one holds without any assumption on the canonical bundle, and even without the Kähler assumption. See [8] for the real analytic case.

Return now to our rational surface S, and suppose that K_S^{-1} admits a smooth metric with semipositive curvature ω. Take a section s of K_S^{-1} vanishing on C, and set

$$F : S \setminus C \longrightarrow \mathbb{R}$$

$$F = -\log \|s\|.$$

Then, on $S \setminus C$, we have $\omega = \mathrm{dd}^c F$. From $\omega \geq 0$ and $\int_S \omega \wedge \omega = c_1^2(S) = 0$, it follows $\omega \wedge \omega \equiv 0$, hence F satisfies the (homogeneous) Monge-Ampère equation:

$$(\mathrm{dd}^c F)^{\wedge 2} \equiv 0.$$

The next lemma provides a second equation satisfied by F. The proof is short, but it is based on a very deep result on the topology of Stein surfaces.

Lemma 3.

$$\mathrm{dd}^c F \wedge \mathrm{d}F \wedge \mathrm{d}^c F \equiv 0.$$

Proof. The function $f = \exp(F)$ is a plurisubharmonic exhaustion of $S \setminus C$, and it is strictly plurisubharmonic precisely on the set where $\mathrm{dd}^c F \wedge \mathrm{d}F \wedge \mathrm{d}^c F$ does not vanish ($\mathrm{dd}^c f = (\mathrm{dd}^c F + \mathrm{d}F \wedge \mathrm{d}^c F) \cdot \exp(F)$). Hence, if that $(2,2)$-form is not identically zero, we deduce from Proposition 2 that $S \setminus C$ is Stein.

Consider now a domain $B = \{F < \lambda\}$, with $\lambda >> 0$. Its boundary M is then diffeomorphic to $\mathbb{T}^3$ (the opposite of the boundary of a tubular neighborhood of C). It is a pseudoconvex boundary, and, since $S \setminus C$ is Stein, it can be smoothly approximated by a *strictly* pseudoconvex boundary $\widehat{M}$, still diffeomorphic to a 3-torus and bounding a Stein domain $\widehat{B}$.

Now, a deep theorem of Stipsicz [**14**] affirms that $\widehat{B}$ must be homeomorphic to $\mathbb{T}^2 \times \mathbb{R}^2$. This is certainly impossible in our case, since $\widehat{B}$ is diffeomorphic to $S \setminus C$ and $S \setminus C$ is not even homotopically equivalent to $\mathbb{T}^2 \times \mathbb{R}^2$ (e.g. look at the intersection form, which is identically zero in the case of $\mathbb{T}^2 \times \mathbb{R}^2$ and non trivial in the case of $S \setminus C$, which contains spheres of selfintersection -2; or, alternatively, look at the Euler characteristic). □

We note that up to now the absence of (-2)-curves is not really indispensable. Indeed, by [**3**] they can be suppressed by a small deformation of the complex structure on a neighborhood of the above $\widehat{B}$ (in other words, Stipsicz's Theorem holds for every relatively compact domain, not necessarily Stein, whose boundary is a strictly pseudoconvex 3-torus).

Remark 4. An interesting problem is about the Steinness of $S \setminus C$ under the sole assumption of absence of compact curves, a special case of a problem raised by Hartshorne, see [**11**, p. 223]. For instance, if C has a pseudoflat neighborhood then $S \setminus C$ is certainly not Stein, due to the presence of compact Levi-flat hypersurfaces, and it is tempting to conjecture that $S \setminus C$ is *never* Stein, whatever the choice of the nine points is. It is not difficult to see that $S \setminus C$ admits a Morse exhaustion function with critical points of index ≤ 2, so that we cannot exclude the Steinness of $S \setminus C$ by "classical" topological arguments (Andreotti-Frankel). On the other side, results by Eliashberg and Gompf [**9**] show that $S \setminus C$ is at least homeomorphic to a Stein surface, and we can even find an open subset $\Omega \subset S \setminus C$ which is Stein and topologically isotopic to $S \setminus C$; the boundary $\partial\Omega$ is a topologically flat 3-torus, topologically isotopic to the boundary of a tubular neighborhood of C. It is worth observing that Stipsicz's Theorem cannot be used here to prove that $S \setminus C$ is never Stein (or, at least, cannot be used in a trivial way). Indeed, if $S \setminus C$ is Stein then we have a strictly plurisubharmonic exhaustion $f : S \setminus C \to \mathbb{R}$, but in principle f may have infinitely many critical points, accumulating to C, and so we cannot ensure that the level sets of f close to C are 3-tori (this is what typically occurs for Stein domains constructed by Eliashberg and Gompf).

The geometrical meaning of Lemma 3 is the following. Set

$$M_\lambda = \{F = \lambda\}$$

and take a regular value λ. Then $T^{\mathbb{C}}M_\lambda$ is spanned by the Kernel of $\mathrm{d}^c F|_{M_\lambda}$, and the vanishing of $\mathrm{dd}^c F \wedge \mathrm{d}F \wedge \mathrm{d}^c F$ is equivalent to the Frobenius integrability of $\mathrm{d}^c F|_{M_\lambda}$ for every λ, i.e. the Levi-flatness of each M_λ. When $\lambda >> 0$, we therefore get a family of Levi-flat 3-tori, approaching to C.

The next step of the proof of Theorem 1 is an elegant cohomological argument from [**5**] to show that these Levi foliations glue together in a holomorphic way (there is also a local counterpart to this fact, see [**2**, Cor. 5.4], which however works only outside the critical points of F).

Without loss of generality, we may assume that the minimum of F on $S \setminus C$ is equal to 0. For every $\lambda > 0$, choose a smooth function φ_λ on $\mathbb{R}$, equal to 1 on $(-\infty, 0]$ and equal to 0 on $[\lambda, +\infty)$. Set

$$F_\lambda = \varphi_\lambda \circ F \in C^\infty_{\mathrm{cpt}}(S \setminus C) \subset C^\infty(S).$$

Then

$$(\mathrm{dd}^c F_\lambda)^{\wedge 2} \equiv 0$$

$$\mathrm{dd}^c F_\lambda \wedge \mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda \equiv 0$$

$$\omega \wedge \mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda \equiv 0$$

as a simple computation shows.

The second equation means that the $(1,1)$-form $\mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda \in A^{1,1}(S)$ is closed (and semipositive). The third equation implies, via the Hodge Index Theorem, that the cohomology class of $\mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda$ is proportional to the class of ω, i.e. to the class of the current δ_C. Hence, by the dd^c-lemma we find an integrable function G_λ on S such that

$$\mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda = c_\lambda \cdot \delta_C + \mathrm{dd}^c G_\lambda$$

for some $c_\lambda \geq 0$. Actually, we have $c_\lambda > 0$ because $\mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda$ is semipositive and not identically zero (for $\lambda > \min F$).

Consider now the open subset of S

$$W_\lambda = \{F > \lambda\} \cup C.$$

It is connected: otherwise, there would be a connected component W'_λ disjoint from C, and the maximum principle applied to $F|_{W'_\lambda}$ would give a contradiction. On this connected open subset the form $\mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda$ is identically zero, hence $\frac{1}{c_\lambda} G_\lambda$ is pluriharmonic on $W_\lambda \setminus C$ and has –logarithmic poles along C. In particular, C has pseudoflat neighborhoods in S, and the first part of Theorem 1 is now completely proved.

The level sets of G_λ close to C are Levi-flat 3-tori, and the corresponding Levi foliations have dense leaves (all isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$); this follows from the fact that N_C is not torsion. A first consequence of this density is that F restricted to such a level set must be constant, i.e. the level sets of G_λ coincide with the hypersurfaces M_μ. A second consequence, in the same spirit, is that $\frac{1}{c_\lambda} G_\lambda$ and $\frac{1}{c_\mu} G_\mu$ differ only by a constant, which can be adjusted to 0 after the choice of some normalization. Set

$$\Sigma = \{F = 0\} \subset S.$$

Because λ is an arbitrary number greater than $0 = \min F$, we finally get:

Proposition 5. *There exists a function*

$$G : S \setminus \Sigma \longrightarrow \mathbb{R} \cup \{+\infty\}$$

which is pluriharmonic outside of C and such that

$$\mathrm{dd}^c G = -\delta_C.$$

This Green function G generates a holomorphic singular foliation $\mathcal{F}$ on $S \setminus \Sigma$, tangent to the Kernel of the 1-form ∂G, which is holomorphic outside of C and has first order poles along C. Of course, $\mathcal{F}$ is tangent to C and nonsingular there. Note that $\mathcal{F}$ is also generated by a holomorphic vector field v on $S \setminus \Sigma$, defined by $i_v \Omega = \partial G$ where Ω is a meromorphic 2-form on S with first order poles on C. This vector field is complete, being tangent to the compact real hypersurfaces M_λ, $\lambda > 0$.

Remark 6. By an index-type argument, it is possible to show that if N_C is totally irrational (i.e. the orbit $\{N_C^{\otimes n}\}_{n\in\mathbb{Z}}$ is dense in $\mathrm{Pic}^\circ(C)$, equivalently the leaves of $\mathcal{F}$ close to C are all isomorphic to $\mathbb{C}$) then the above vector field v has no singularities at all. Hence, in that case, *every* M_λ, $\lambda > 0$, is a smooth 3-torus, and $S \setminus C$ retracts (topologically) onto Σ. Moreover, G continuously extends to Σ, as a constant.

It is at this point that the real analyticity assumption enters in a massive way:

Lemma 7. *If F is real analytic, then the foliation $\mathcal{F}$ extends to a holomorphic foliation on the full surface S.*

Proof. The subset $\Sigma \subset S$ is real analytic, of dimension at most 3. It is easy to find a real analytic subset $\Sigma_1 \subset \Sigma$ of dimension at most 2 such that every connected component of $\Sigma_0 = \Sigma \setminus \Sigma_1$ is a smooth hypersurface (or stratum) of dimension 3 which is either strictly pseudoconvex or Levi flat.

If $\Sigma_0' \subset \Sigma_0$ is a strictly pseudoconvex stratum, then $\mathcal{F}$ (and also the holomorphic 1-form ∂G) extends through Σ_0', by Hartogs. If instead Σ_0' is a Levi-flat stratum, then we have on Σ_0' the Levi foliation, and we need only to prove that this Levi foliation glues in a real analytic way to the foliation $\mathcal{F}$ outside of Σ_0'. But this is a rather trivial fact: $\mathcal{F}$ is tangent to the Kernel of $\mathrm{d}F \wedge \mathrm{d}^c F$, and around any point of Σ_0' we can divide $\mathrm{d}F \wedge \mathrm{d}^c F$ by some power of an equation ϱ of Σ_0' so that we get a real analytic $(1,1)$-form $\left(\frac{1}{\varrho^k}\right)\mathrm{d}F \wedge \mathrm{d}^c F$ which is no more identically zero along Σ_0' (put in another way: the "essential" singularities of a real analytic subsheaf of the tangent bundle of a real analytic manifold always have codimension at least 2). Actually, this argument applies also to the strictly pseudoconvex strata.

In this way, we have extended $\mathcal{F}$ to $S \setminus \Sigma_1$, and $\dim \Sigma_1 \leq 2$. Each irreducible component of Σ_1 is either a point, or a real curve, or a real surface which is totally real at a generic point (absence of compact complex curves in $S \setminus C$). In all cases, Hartogs theorem provides the extendibility of $\mathcal{F}$ through Σ_1. □

As in Lemma 3, the absence of (-2)-curves in $S \setminus C$ is not really important here, since a foliation outside a compact complex curve of negative selfintersection extends through the curve.

Remark that, by the above proof, the only obstruction to the holomorphic extension of ∂G (or of v) is represented by the Levi-flat strata. Informally, one may think that, in the real analytic case, the Levi-flat pieces of Σ support a (foliated) closed positive current T, and that G extends to an integrable function, still denoted by G, which realizes a cobordism between δ_C and T:

$$\mathrm{dd}^c G = T - \delta_C.$$

This current T is a weak limit of the smooth currents $\frac{1}{c_\lambda} \mathrm{d}F_\lambda \wedge \mathrm{d}^c F_\lambda$. Actually, a similar interpretation holds even if F is only smooth, and not real analytic, the current T being supported in the boundary of Σ, and presumably "laminated". However, in the smooth case this subset Σ could be very large, for instance it could have a nonvoid interior, and then we do not know how to extend the foliation through it. Roughly speaking, the complement of Σ is the maximal pseudoflat neighbourhood of C, and one may well imagine that it may be small (but we do not know any concrete example).

The last step of the proof of Theorem 1 is an argument from the general Kodaira-dimension-type classification of foliations [**10**] [**4**]. Instead of referring to general results, we prefer to give a simple and direct proof of the result that we need.

Proposition 8. *There does not exist a holomorphic foliation $\mathcal{F}$ on S which is tangent to C and free of singularities along C.*

Proof. Let $K_{\mathcal{F}}$ be the canonical bundle of an hypothetical foliation $\mathcal{F}$. By Miyaoka's Theorem [**4**, p. 56], $K_{\mathcal{F}}$ is certainly pseudoeffective, because $\mathcal{F}$ is certainly not a foliation by rational curves. Moreover, since $S \setminus C$ contains no compact complex curves, we see that $K_{\mathcal{F}}$ is even nef [**4**, p. 57] (a curve D with $c_1(K_{\mathcal{F}}) \cdot D < 0$ would be necessarily $\mathcal{F}$-invariant, hence contained in $S \setminus C$). By the Hodge Index Theorem, and $c_1(K_{\mathcal{F}}) \cdot C = 0$, we then get that $K_{\mathcal{F}}$ is proportional to $\mathcal{O}_S(C)$, and in particular:

$$c_1^2(K_{\mathcal{F}}) = c_1(S) \cdot c_1(K_{\mathcal{F}}) = 0.$$

Take now the conormal bundle $N^*_{\mathcal{F}}$ of $\mathcal{F}$. By adjunction ($K_S = K_{\mathcal{F}} \otimes N^*_{\mathcal{F}}$) we have $h^1(K_{\mathcal{F}}) = h^1(N^*_{\mathcal{F}})$ and $h^2(K_{\mathcal{F}}) = h^0(N^*_{\mathcal{F}}) = 0$, the latter equality because there are no global holomorphic 1-forms on S. Hence the Riemann-Roch formula gives:

$$h^0(K_{\mathcal{F}}) = h^1(N^*_{\mathcal{F}}) + \chi(\mathcal{O}_S) = h^1(N^*_{\mathcal{F}}) + 1.$$

However, $h^0(K_{\mathcal{F}}) \leq 1$ because otherwise we get an elliptic fibration tangent to C, and therefore:

$$h^1(N^*_{\mathcal{F}}) = 0.$$

Consider now the exact sequence

$$0 \to N^*_{\mathcal{F}} \to N^*_{\mathcal{F}} \otimes \mathcal{O}_S(C) \to N^*_{\mathcal{F}} \otimes \mathcal{O}_S(C)|_C \to 0$$

and note that $N^*_{\mathcal{F}} \otimes \mathcal{O}_S(C)|_C \simeq \mathcal{O}_C$ (the isomorphism being the Residue Map: the fact that $\mathcal{F}$ is tangent to C precisely means that $N^*_{\mathcal{F}} \otimes \mathcal{O}_S(C) \subset \Omega^1_S(\log C)$). By the above h^1-vanishing, we get a nontrivial global section α of $N^*_{\mathcal{F}} \otimes \mathcal{O}_S(C)$. This is impossible for at least two reasons:

(i) $\mathrm{Res}(\alpha) = a[C]$ for some $a \neq 0$, contradicting the fact that the (total) residue of a logarithmic form must be cohomologous to zero ($\mathrm{Res}(\alpha) = \overline{\partial}\alpha$).

(ii) By adjunction and $K_S^{-1} = \mathcal{O}_S(C)$ we get also a global section of $T_{\mathcal{F}} = K_{\mathcal{F}}^{-1}$, nonvanishing on C, i.e. a global holomorphic vector field on S with an elliptic orbit. As is well-known, this is impossible on a rational surface. □

This completes the proof of Theorem 1.

Remark 9. There is a version of the previous proposition (which can be extracted from [**10**] or [**4**]) which does not require the absence of (-2)-curves in $S \setminus C$. The conclusion, however, is not the inexistence of the foliation, but the statement that $\mathcal{F}$ is a quotient of a foliation on a ruled surface with elliptic base, transverse to the fibers (this discrepancy is due to the fact that, when there are (-2)-curves, the pseudoeffective $K_{\mathcal{F}}$ may have a nontrivial negative part, which can be eliminated by a covering, but this operation destroys the rationality of the surface). See [**11**, p. 224] for explicit examples, and [**13**] for a thorough analysis of the coverings that may occur. In particular, Ogus' examples show that for special choices of C_0 and $p_1, \ldots, p_9$ one can obtain a nonelliptic surface S which admits a real analytic Kähler metric with semipositive curvature.

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