

# Differential geometry exam (May 20, 2013)

**Rules:** Each student should count his or her handout score (as explained in the handouts rules page) and give me the score sheets with the scores marked.

Each examinee receives a set of 12 problems to solve. The solutions should be explained to examiners orally.

The points for the exam are computed by summing up points for the problems. Results will be announced at <http://bogomolov-lab.ru/KURSY/GEOM-2013/>

The final score is determined (for HSE purposes) as  $2 + [0.1b]$ , where  $b$  is total of points for handouts, exam and two test assignments.

## 1.1 Hausdorff dimension

**Exercise 1.1 (5).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be any function, and  $\Gamma_f \subset \mathbb{R} \times \mathbb{R}$  its graph. Prove that the Hausdorff dimension of  $\Gamma_f$  is  $\geq 1$ .

**Exercise 1.2 (10).** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function, and  $\Gamma_f \subset \mathbb{R} \times \mathbb{R}$  its graph. Find the Hausdorff dimension of  $\Gamma_f$ .

**Exercise 1.3 (15).** Construct a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that the Hausdorff dimension of its graph  $\Gamma_f \subset \mathbb{R}^2$  is bigger than 1.

**Exercise 1.4 (10).** Construct a function  $f : [0, 1] \rightarrow [0, 1]$  which is continuous, but not Lipschitz on any interval  $[a, b] \subset [0, 1]$ .

## 1.2 Geometry of manifolds

**Exercise 1.5 (5).** Prove that any compact manifold is metrizable.

**Exercise 1.6 (10).** Let  $M = \bigcup_i M_i$  be a union of countably many compact manifolds  $M_i$  with boundary,  $M_1 \subset M_2 \subset M_3 \subset \dots$ . Prove that  $M$  is metrizable.

**Exercise 1.7 (5).** Let  $V \subset U \subset M$  be open subsets of a metric space  $M$ , and the closure of  $V$  is contained in  $U$ . Prove that there exists a continuous function  $M \rightarrow [0, 1]$  which is equal to 1 on  $V$  and 0 outside of  $U$ .

**Exercise 1.8 (10).** Let  $M$  be a metric space. Suppose that  $M$  is equipped with locally finite covers  $U_\alpha$  and  $V_\alpha$ , indexed by the same set, and the closure of  $V_\alpha$  is contained in  $U_\alpha$  for each  $\alpha$ . Construct a set of continuous functions  $\phi_\alpha : M \rightarrow [0, 1]$  such that  $\phi_\alpha = 0$  outside of  $U_\alpha$ ,  $\phi_\alpha = 1$  on  $V_\alpha$ , and  $\sum_\alpha \phi_\alpha \geq 1$ .

**Exercise 1.9 (10).** Let  $M$  and  $M'$  be smooth manifolds, such that the ring  $C^\infty M$  is isomorphic to  $C^\infty M'$ . Prove that  $M$  is diffeomorphic to  $M'$ .

## 1.3 Germs of functions

**Exercise 1.10 (5).** Let  $R$  be a ring of germs of continuous functions on  $\mathbb{R}$  in 0, and  $I \subset R$  an ideal. Consider the subset  $I_+ \subset I$  of all non-negative functions in  $I$ . Prove that  $I$  is generated by  $I_+$ , or find a counterexample.

**Exercise 1.11 (5).** Let  $R$  be a ring of germs of continuous functions on  $\mathbb{R}$  in 0, and  $I \subset R$  an ideal. Consider the subset  $I_s \subset I$  of all smooth functions in  $I$ . Prove that  $I$  is generated by  $I_s$ , or find a counterexample.

**Exercise 1.12 (5).** Let  $R$  be a ring of germs of smooth functions on  $\mathbb{R}^n$  in 0, and  $K \subset R$  – intersection of all powers of maximal ideal. Prove that  $K$  is generated by all non-negative  $\phi \in K$ .

**Exercise 1.13 (15).** Let  $R$  be a ring of continuous  $\mathbb{R}$ -valued functions on a topological space  $M$ , and  $I \subset R$  a prime ideal. Prove that  $I^2 = I$ .

**Exercise 1.14 (5).** Find all prime ideals in a ring of germs of continuous functions on  $\mathbb{R}$  in 0.

## 1.4 Topology of manifolds and vector bundles

**Exercise 1.15 (5).** Let  $M$  be a smooth manifold, and  $TM$  the total space of its tangent bundle. Prove that  $TM$  is orientable, or find a counterexample.

**Exercise 1.16 (5).** Let  $B$  be a 1-dimensional real vector bundle on a smooth manifold  $M$ . Show that  $B \otimes B$  is a trivial bundle.

**Exercise 1.17 (10).** Let  $B$  be a non-trivial vector bundle on a smooth manifold  $M$ , and  $B_1 := B \oplus C^\infty M$  a direct sum of  $B$  and a trivial 1-dimensional bundle. Find a non-trivial bundle  $B$  such that  $B_1$  is trivial.

**Exercise 1.18 (15).** Let  $M$  be a compact, even-dimensional manifold. Prove that  $M$  admits a non-trivial vector bundle.

**Exercise 1.19 (10).** Let  $M = S^{2n}$  be an even-dimensional sphere. Prove that  $TM$  does not admit a metric of signature  $(1, 2n - 1)$ .

## 1.5 Differential forms and differential operators

**Exercise 1.20 (10).** Let  $M$  be a smooth manifold. Denote by  $L_a \in \text{End}(C^\infty M)$  an operator of multiplication by  $a \in C^\infty M$ . A **differential operator of first order** is a map  $D : C^\infty M \rightarrow C^\infty M$  which satisfies  $[D, L_a](z) = z[D, L_a](1)$  for each  $a \in C^\infty M$ . Prove that the differential operators of first order are **local**, that is, for each  $f \in C^\infty M$  and an open set  $U \subset M$ , the functions  $D(f)|_U$  is determined by  $f|_U$ .

**Exercise 1.21 (5).** Let  $\alpha$  be a closed differential form with compact support on  $\mathbb{R}^n$ . Prove that there exists a  $k$ -form  $\beta$  with support in unit ball such that  $\alpha - \beta = d\gamma$ , where  $\gamma$  is a form with compact support.

**Exercise 1.22 (10).** Let  $\delta : \Lambda^* M \rightarrow \Lambda^* M$  be a derivation of de Rham algebra, and  $\alpha$  a differential form with support in a closed set  $S \subset M$ . Prove that  $\delta(\alpha)$  has support in  $S$ .

**Exercise 1.23 (15).** Let  $t \in \Lambda^2 TM$  be a skew-symmetric bivector on a manifold  $M$ , and  $i_t : \Lambda^i M \rightarrow \Lambda^{i-2} M$  an operation of substitution a bivector into an  $i$ -form. Consider a  $k$ -form  $\eta$ , and let  $e_\eta(v) = \eta \wedge v$  be the corresponding map on de Rham algebra,  $e_\eta : \Lambda^i M \rightarrow \Lambda^{i+k}(M)$ . Prove that the commutator  $[e_\eta, i_t]$  is a derivation.

## 1.6 Fibered spaces and fibrations

**Definition 1.1.** Recall that  $U(n)$  is a group of all unitary automorphisms of  $\mathbb{C}^n$ ,  $SO(n)$  a group of all orthogonal, oriented automorphisms of  $\mathbb{R}^n$ , and  $Sp(n)$  a group of all quaternionic linear, unitary automorphisms of  $\mathbb{H}^n$ . These groups are called **classical Lie groups**.

**Exercise 1.24 (10).** Let  $G$  be a classical Lie group. Find a fibration with a total space  $G$ , base  $S^n$  (a sphere), and fiber  $G_0$ , which is also a Lie group. Find  $G_0$  in each of these cases.

**Exercise 1.25 (10).** Compute the real dimension of each classical Lie group.

**Exercise 1.26 (5).** Construct a fibration with total space  $SU(3)$ , base  $S^5$  and fiber  $S^3$ .

**Exercise 1.27 (5).** Construct a non-trivial fibration with fiber  $S^3$ .